A MIXED BOUNDARY VALUE PROBLEM FOR AN ANNULUS

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Among ring domains, i.e. doubly-connected domains bounded by two continua, an annulus plays often a particular role of canonical domain. Several functions, analytic or harmonic in an annulus, satisfying some types of preassigned boundary conditions can be expressed within a range of elementary or elliptic functions. Especially, the formulas due to Villat and Dini [1] for Dirichlet problem belong to this category, and a formula for Neumann problem does also [2].

Once Tomotika[3] has derived an expression of such a type for a related problem which has been shown to be useful for solving several problems on hydro- and aerodynamics. The problem may be stated as follows:

To determine an analytic function $f(z)$ regular single-valued and bounded in an annulus $q<|z|<1$ and satisfying the boundary conditions

$$\Re f(e^{i\varphi}) = \Phi(\varphi)$$

and

$$\Im f(qe^{i\varphi}) = 0$$

for

$$0 \leqq \varphi < 2\pi.$$

In order to derive an expression for the solution, he followed faithfully first the Villat's method and then the Demotchenko's in his respective papers cited in [3]. However, this problem can be immediately reduced to a Dirichlet problem after an analytic prolongation by inversion and then solved quite briefly. In fact, the problem is evidently equivalent to determine an analytic function $f(z)$ regular single-valued and bounded in the duplicated annulus $q^2<|z|<1$ and satisfying the boundary conditions

$$\Re f(e^{i\varphi}) = \Re f(q^2 e^{i\varphi}) = \Phi(\varphi)$$
$$\text{for} \quad 0 \leqq \varphi < 2\pi,$$

an arbitrary purely imaginary constant which is additively involved in the solution is to be determined by $\Im f(q) = 0$; since $\Im f(z)$ remains constant along $|z| = q$, it then vanishes out along $|z| = q$.

Now, in the last problem, the condition for single-valuedness (monodromy condition) is surely satisfied:

$$\int_0^{2\pi} \Re f(e^{i\varphi})\, d\varphi = \int_0^{2\pi} \Re f(qe^{i\varphi})\, d\varphi.$$

Consequently, by means of Villat's formula, the solution is obtained in the form

$$f(z) = \frac{\hat{\omega}_1}{i\pi^2}\int_0^{2\pi} \Phi(\varphi) \cdot \left\{ \hat{\zeta}\left(\frac{\hat{\omega}_1}{\pi}(i\lg z+\varphi)\right) - \hat{\zeta}_3\left(\frac{\hat{\omega}_1}{\pi}(i\lg z+\varphi)\right)\right\} d\varphi,$$

in conformity with the formula of Tomotika, where the notations for Weierstrassian theory of elliptic functions, marked by ^ , refer to those with primitive periods $2\hat{\omega}_1$ and $2\hat{\omega}_3$ satisfying a relation

$$\hat{\omega}_3 / \hat{\omega}_1 = -2i\lg q/\pi;$$

an additive constant has been adjusted so as $\Im f(q) = 0$.

In the present Note, we shall deal with a slightly general type of mixed boundary value problem by means of a similar method as explained above. It is formulated as follows:

To determine a function $u(z)$, $z = re^{i\theta}$, harmonic and bounded in an annulus $q<|z|<1$ and satisfying the boundary conditions

$$u(qe^{i\varphi}) = N(\varphi)$$

and