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FIXED POINTS FOR CONDENSING MULTIFUNCTIONS IN METRIC SPACES WITH CONVEX STRUCTURE

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In this paper, we prove a fixed point theorem for condensing multifunctions with convex values, closed graph, and bounded range acting on a metric space endowed with a simple but powerful notion of convexity.

In Section 1, we discuss a notion of convexity for metric spaces which was introduced in [8] by W. Takahashi. We develop here some geometric and topological properties which result when a uniqueness assertion is added to Takahashi's requirements.

In Section 2, we introduce a new notion of convex structure for a metric space. This new notion is based on the Takahashi notion, but has some pleasanter geometric properties, which we investigate here. In particular, we are here able to permute the order of repeated convex combination.

In Section 3, we introduce the "measure of noncompactness" and study it in relation to a "stable" convex structure. The major fact here is that the measure of non-compactness is invariant under passage to convex hulls.

Section 4 is devoted to our major result (Theorem 4.2): A condensing multifunction with convex values, closed graph, and bounded range, which acts on a complete metric space with stable strong convex structure has a fixed point.

1. Takahashi convex structures.

1.1 DEFINITION: Let (X, d) be a metric space, and let I be the closed unit interval $[0, 1]$. A *Takahashi convex structure* (TCS) on X is a function $W: X\times X\times I\rightarrow X$ which has the property that for every $x, y\in X$ and $t\in I$ we have

$$d(z, W(x, y, t))\leqq td(z, x)+(1-t)d(z, y) \tag{1}$$

for every $z\in X$. If (X, d) is equipped with a TCS, we call X a *convex metric space*. When (X, d) is a convex metric space and $S\subset X$, we say that S is *convex* provided that $W(x, y, t)$ lies in S for each (x, y, t) in $S\times S\times I$.

Takahashi convex structures were introduced by W. Takahashi in [8], and

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