ON A CERTAIN SEQUENCE OF CHANCE VARIABLES.

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1. Let $\{x_n\}$ be a sequence of chance variables, each of which has an expectation $E(x_n)$, satisfying the following condition;

(F) $\qquad E_m(x_n) = x_m \qquad (m \leqq n)$

with probability 1, where $E_m(x_n)$ denotes the conditional expectation of $x_n$ for given $x_1, x_2, ----, x_m$[1]. In the present note, we shall give the sufficient conditions for the strong law of the large number and the central limit theorem in such a sequence of chance variables.

Theorem 1. Let $\{x_n\}$ be a sequence of chance variables satisfying the condition (F). Then the convergence of the following series

$$\sum_{k=0}^{\infty} \frac{1}{2^k} E(|x_{2^{k+1}} - x_{2^k}|)$$

is sufficient for the strong law of the large number, that is,

$$P_r\left\{\lim_{n\to\infty} \frac{x_n}{n} = 0\right\} = 1.$$

Proof. Let $E_m^{(k)}$ denote the set

$$\{x_{2^k+1} - x_{2^k} \leqq \varepsilon 2^k, ---, x_{m-1} - x_{2^k} \leqq \varepsilon 2^k, x_m - x_{2^k} > \varepsilon 2^k\}$$

for any $\varepsilon > 0$ and a positive integer $m$ such that $2^k < m \leqq 2^{k+1}$. It is evident that $E_i^{(k)}$ and $E_j^{(k)}$ $(i \neq j)$ are disjoint;

(1) $\qquad E_i^{(k)} \cdot E_j^{(k)} = 0 \qquad (i \neq j)$

From the definition of the conditional expectation and the condition (F), we have

$$(2) \int_{E_m^{(k)}} (x_{2^{k+1}} - x_{2^k}) dP = \int_{E_m^{(k)}} (x_m - x_{2^k}) dP \geqq \varepsilon 2^k P_r\{E_m^{(k)}\}$$

Putting

$$E^{(k)} = \sum_{m=2^k+1}^{2^{k+1}} E_m^{(k)},$$

from (1) and (2), we obtain

$$\int_{E^{(k)}} (x_{2^{k+1}} - x_{2^k}) dP \geqq \varepsilon 2^k P_r\{E^{(k)}\}$$

and, a posteriori,

$$E(|x_{2^{k+1}} - x_{2^k}|) \geqq \varepsilon 2^k P_r\{E^{(k)}\}$$

Hence the assumption of the theorem implies the convergence of the series $\sum_{k=0}^{\infty} P_r\{E^{(k)}\}$. It follows that, by the Borel-Canteli's theorem, for sufficiently large $k$ and $2^{k+1} \geqq m > 2^k$, the inequality $x_m - x_{2^k} < \varepsilon 2^k$ holds with the probability 1. Denoting the integral part of $\log n/\log 2$ by $p$, for an arbitrary $n > 2^k$, we have

$$x_n - x_{2^k} = (x_n - x_{2^p}) + (x_{2^p} - x_{2^{p-1}}) + \cdots + (x_{2^{k+1}} - x_{2^k})$$
$$< \varepsilon (2^p + \cdots + 2^k) < \varepsilon 2^{p+1}$$

For a fixed $k$, let $n$ tend to $\infty$, then

$$\overline{\lim} \frac{x_n}{n} \leqq 2\varepsilon.$$

$\varepsilon$ being an arbitrary positive number, it follows that

$$\overline{\lim} \frac{x_n}{n} \leqq 0.$$

In the same way, we obtain

$$\underline{\lim} \frac{x_n}{n} \geqq 0$$

and hence $\lim \frac{x_n}{n} = 0$

Thus theorem is proved.