NOTE ON 3-FACTOR SETS.

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Recent increasing concern in higher dimensional factor sets, in connection with higher dimensional cohomology theory in rings and groups[1], encourages the writer to offer here a small result which he obtained in succession to Teichmuller's work[2], which he, however, did not publish because of its some immatuarity. It is to raise dimensions by 1 in his previous study on relationship between usual 2-factor sets and norm class group[3].

Let K be a Galois extension over a field F, and $G = \{1, \lambda, \mu, \nu, \cdots, \pi\}$ be its Galois group. A system $\{a_{\lambda,\mu,\nu}\}$ of $g^3 = (G)^3$ non-zero elements in K is called a 3-factor set, if

$$(0) \quad a_{\lambda,\mu,\nu} \cdot a_{\lambda,\mu\nu,\pi} \cdot a^{\lambda}_{\mu,\nu,\pi} = a_{\lambda,\mu,\nu\pi}\, a_{\lambda\mu,\nu,\pi}$$

for every λ, μ, ν, π.

We introduce a simplified notation to denote $\prod_{\lambda\in G} a_{\lambda,\mu,\nu}$, for instance, by $a_{G,\mu,\nu}$, and similarly to denote $\prod_{\lambda\in G} a^{\lambda}_{\mu,\nu,\pi} = N_{K/F}(a_{\mu,\nu,\pi})$ by $a^{G}_{\mu,\nu,\pi}$. Then we have, from (0),

$$(1) \quad a_{G,\mu,\nu}\, a_{G,\mu\nu,\pi}\, a^{G}_{\mu,\nu,\pi} = a_{G,\mu,\nu\pi}\, a_{G,\nu,\pi}$$

$$(2) \quad a_{\lambda,G,\nu}\, a_{\lambda,G,\pi}\, a^{\lambda}_{G,\nu,\pi} = a_{\lambda,G,\nu\pi}\, a_{G,\nu,\pi}$$

$$(3) \quad a_{\lambda,\mu,G}\, a_{\lambda,G,\pi} . a^{\lambda}_{\mu,G,\pi} = a_{\lambda,\mu,G}\, a_{\lambda\mu,G,\pi}$$

$$(4) \quad a^{G}_{\lambda,\mu,\nu}\, a_{\lambda,\mu\nu,G}\, a^{\lambda}_{\mu,\nu,G} = a_{\lambda,\mu,G}\, a_{\lambda\mu,\nu,G}.$$

(3) gives

$$(3') \quad a_{\lambda,G,\pi}\, a^{\lambda}_{\mu,G,\pi} = a_{\lambda\mu,G,\pi}$$

There exists therefore, by virtue of the theorem of so-called transformation elements, an element b_{π} $(\neq 0)$ for each π, such that

$$(5) \quad a_{\lambda,G,\pi} = b_{\pi}^{1-\lambda}$$

Hence $N_{K/F}(a_{\lambda,G,\pi}) = 1$, which can, of course, be deduced directly from (3') too. Further, (2) may be written as

$$(2') \quad \frac{a_{\lambda,G,\nu}\, a_{\lambda,G,\pi}}{a_{\lambda,G,\nu\pi}} = a^{1-\lambda}_{G,\nu,\pi}$$

On combining with (5) we have

$$(6) \quad \left(\frac{b_{\nu}\, b_{\pi}}{b_{\nu\pi}}\right)^{1-\lambda} = a^{1-\lambda}_{G,\nu,\pi}$$

and so the elements

$$\alpha_{\mu,\nu} = \frac{b_{\mu} b_{\nu}}{b_{\mu\nu}}\, a^{-1}_{G,\mu,\nu}$$

are contained in the ground field F. Moreover, the system $\{\alpha_{\mu,\nu}\}$ forms a (usual 2-) factor set mod. the norm group $N^{*}_{K/F}$; $N^{*}_{K/F}$ consisting of the totality of the norms of non-zero elements of K with respect to F. For, (1) gives

$$(7) \quad a_{G,\mu,\nu}\, a_{G,\mu\nu,\pi} = a_{G,\mu,\nu\pi}\, a_{G,\nu,\pi} \ \mathrm{mod.}\ N^{*}_{K/F}$$

while trivially

$$\left(\frac{b_{\mu} b_{\nu}}{b_{\mu\nu}}\right)\left(\frac{b_{\mu\nu} b_{\pi}}{b_{\mu\nu\pi}}\right) = \left(\frac{b_{\mu} b_{\nu\pi}}{b_{\mu\nu\pi}}\right)\left(\frac{b_{\nu} b_{\pi}}{b_{\nu\pi}}\right),$$

and thus

$$(8) \quad \alpha_{\mu,\nu}\, \alpha_{\mu\nu,\pi} \equiv \alpha_{\mu,\nu\pi}\, \alpha_{\nu,\pi} \ (= \alpha_{\mu,\nu\pi}\, \alpha^{\mu}_{\nu,\pi}) \quad \mathrm{mod.}\ N^{*}_{K/F}$$

The associate class of the system is uniquely determined, up to mod. $N^{*}_{K/F}$, by the associate class of $\{a_{\lambda,\mu,\nu}\}$. To prove this, we have first to show that the class of $\{\alpha_{\mu,\nu}$ mod. $N^{*}_{K/F}\}$ is independent of the choice of $\{b_{\mu}\}$. But a different choice may be given by $\{\beta b_{\mu}\}$ with $\beta\ (\neq 0) \in F$. Then $\{\alpha_{\mu,\nu}\}$ is replaced, correspondingly, by $\{\beta\alpha_{\mu,\nu}\}$, which gives certainly a system associate to $\{\alpha_{\mu,\nu}\}$. Consider further a system $\{a'_{\lambda,\mu,\nu}\}$ associate to our $\{a_{\lambda,\mu,\nu}\}$. It is given as

$$a'_{\lambda,\mu,\nu} = a_{\lambda,\mu,\nu}\, \frac{a_{\lambda,\mu}\, a_{\lambda\mu,\nu}}{a^{\lambda}_{\mu,\nu}\, a_{\lambda,\mu\nu}}$$

Hence

$$a'_{\lambda,G,\nu} = a_{\lambda,G,\nu}\, \frac{a_{\lambda,G}\, a_{G,\nu}}{a^{\lambda}_{G,\mu}\, a_{\lambda,G}} = a_{\lambda,G,\nu}\, a^{1-\lambda}_{G,\nu}$$

So we may adopt as b'_{ν}, corresponding to our $\{a'\}$, $b'_{\nu} = b_{\nu}\, a_{G,\nu}$. Then

$$\alpha'_{\mu,\nu} = \frac{b'_{\mu} b'_{\nu}}{b'_{\mu\nu}} (a'_{G,\mu,\nu})^{-1} = \frac{b_{\mu} b_{\nu}}{b_{\mu\nu}}\, \frac{a_{G,\mu}\, a_{G,\nu}}{a_{G,\mu\nu}}\, a^{-1}_{G,\mu,\nu} \cdot$$

$$\cdot \frac{a^{G}_{\mu,\nu}\, a_{G,\mu\nu}}{a_{G,\mu}\, a_{G,\nu}} = \frac{b_{\mu} b_{\nu}}{b_{\mu\nu}}\, a^{-1}_{G,\mu,\nu}\, a^{G}_{\mu,\nu} = \alpha_{\mu,\nu}\, a^{G}_{\mu,\nu} \equiv \alpha_{\mu,\nu} \quad \mathrm{mod.}\ N^{*}_{K/F}$$

The uniqueness assertion of the class of $\{\alpha_{\mu,\nu}\}$ mod. $N^{*}_{K/F}$ is thus proved. It is further readily seen that a product of two (3-)factor sets corresponds to the