ON THE EXISTENCE OF ANALYTIC FUNCTIONS ON CLOSED ANALYTIC SURFACES.

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1. Introduction. The present short note is a preliminary report on an attempt to generalize the classical existence theorem of analytic functions on closed Riemann surfaces[1] to the case of the theory of functions of two complex variables. Let $\mathfrak{M}$ be a closed analytic surface, i.e. a 2-dimensional (topologically 4-dimensional) analytic manifold; the local (analytic) coordinates on $\mathfrak{M}$ will be denoted by z^1, z^2. The poles and zero-points of a one-valued meromorphic function $f(z^1, z^2)$ defined on $\mathfrak{M}$ constitute a 1-dimensional analytic submanifold of $\mathfrak{M}$ consisting of a finite number of irreducible closed analytic curves $\Gamma_1, \Gamma_2, \cdots, \Gamma_k, \cdots\cdots$, each of which is a polar or a zero-point curve of $f(z^1, z^2)$ The formal sum

$$D = \sum m_k \Gamma_k$$

of these curves multiplied respectively by the multiplicity m_k of Γ_k is called the divisor of $f(z^1, z^2)$, where the multiplicities of the polar curves are to be associated with the negative sign. The divisor D of $f(z^1, z^2)$ can be also defined in case $f(z^1, z^2)$ is a many-valued meromorphic function, if the absolute value $|f(z^1, z^2)|$ is one-valued on $\mathfrak{M}$. Such a function is called multiplicative, since, if one prolongs $f(z^1, z^2)$ analytically along a closed continuous curve γ, then $f(z^1, z^2)$ is multiplied by a constant factor $\chi(\gamma)$ depending only on the homology class of γ on $\mathfrak{M}$. From the topological viewpoint, the divisor D is a 2-cycle on $\mathfrak{M}$. It can be readily proved that the divisor D of an arbitrary multiplicative meromorphic function on $\mathfrak{M}$ satisfies

$D \approx 0$ (homology with division allowed).

Assume now that a cycle $D = \sum m_k \Gamma_k$ consisting of a finite number of irreducible closed analytic curves $\Gamma_1, \Gamma_2, \cdots\cdots$ is given. Then, does a multiplicative meromorphic function $f(z^1, z^2)$ on $\mathfrak{M}$ having D as its divisor exist? In what follows this fundamental question will be answered affirmatively under the assumption that a positive definite Hermitian metric

$$ds^2 = g_{\alpha\bar{\beta}}\, dz^\alpha d\bar{z}^\beta$$

without torsion is defined on $\mathfrak{M}$. In this note we shall give the main results and the outline of the proofs only. The detailed proofs and more systematic theory of analytic functions on analytic surfaces will be given elsewhere.

§2. Harmonic integrals.

Putting

$$z^1 = x^1 + \sqrt{-1}\, x^2, \qquad z^2 = x^3 + \sqrt{-1}\, x^4,$$

we introduce the real coordinates x^1, x^2, x^3, x^4 on $\mathfrak{M}$. Then $\mathfrak{M}$ becomes a 4-dimensional closed Riemannian manifold with the positive definite metric

$$ds^2 = 2 g_{\alpha\bar{\beta}}\, dz^\alpha d\bar{z}^\beta = g_{jk}\, dx^j dx^k$$

(in what follows Latin subscripts j, k etc. take values ranging from 1 to 4 and Greek subscripts α, β denote 1 or 2).

Now we shall consider differential forms

$$\psi = \psi^p = \frac{1}{p!}\psi_{jk\cdots l}[dx^j dx^k \cdots dx^l]$$

defined on $\mathfrak{M}$, where p denotes the rank of ψ. The form ψ is said to be measurable, to have continuous derivatives or to be regular, if the coefficients $\psi_{jk\cdots l}$ are measurable, have continuous derivatives or regular analytic as functions of the local coordinates x^1, x^2, x^3, x^4. The derived form and the dual form of ψ will be denoted by $\mathfrak{r}^*\psi$ and $\mathfrak{D}\psi$, resp.; as is well known, they are defined by

$$\mathfrak{r}^*\psi^p = \frac{1}{p!}[d\psi_{jk\cdots l}\, dx^j dx^k \cdots dx^l],$$

$$\mathfrak{D}\psi^p = \frac{1}{p!(4-p)!}\frac{1}{\sqrt{g}}\,\mathrm{sgn}\begin{pmatrix} i\cdots j\, k\cdots l \\ 1\, 2\, \cdots\cdots 4\end{pmatrix}$$

$$\times\, g_{ip}\cdots g_{jq}\psi_{k\cdots l}[dx^p\cdots dx^q],$$

where $g = \det(g_{jk})$. The dual derivation $\mathfrak{r}$ and the Laplacian Δ are defined by

$$\mathfrak{r} = \mathfrak{D}\mathfrak{r}^*\mathfrak{D},$$

$$\Delta = -\mathfrak{r}\mathfrak{r}^* - \mathfrak{r}^*\mathfrak{r}.$$