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DISCRETE MEASURES AND THE RIEMANN HYPOTHESIS

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1. Introduction

The purpose of this paper is to show that the Riemann hypothesis is equivalent to a problem of the rate of convergence of certain discrete measures defined on the positive real numbers to the measure $\frac{6}{\pi^2}udu$, where du is Lebesgue measure.

As a motivation consider the following: For each positive real number y, let μ_y be the infinite measure on the real line defined by

$$\mu_y = \sum_{n\in\mathbb{Z}} y\,\delta_{ny},$$

where $\mathbb{Z}$ denotes the integers and δ_x denotes the Dirac mass at the point $x \in \mathbb{R}$. It follows by the Poisson summation formula that if $f \in C_c^\infty(\mathbb{R})$ ($C_c^\infty(\mathbb{R})$ = functions $f : \mathbb{R} \to \mathbb{R}$, of class C^∞ and with compact support), then for every $\beta > 0$:

$$\mu_y(f) = \int_{\mathbb{R}} f(t)\,dt + o(y^\beta), \qquad \text{as } y \to 0.$$

This is so because by the Poisson summation formula [B],

$$y\sum_{n\in\mathbb{Z}} f(ny) = \sum_{n\in\mathbb{Z}} \widehat{f}(ny^{-1})$$

where $\widehat{f}$ is the Fourier transform of f and, since f is smooth with compact support we have that $\widehat{f}$ is of rapid decay at infinity. Hence

$$y\sum_{n\in\mathbb{Z}} f(ny) = \widehat{f}(0) + o(y^\beta) \quad \text{as } y \to 0 \text{ for all } \beta > 0.$$

So, as $y \to 0$, the atoms of μ_y cluster uniformly and $\mu_y(f)$ gives a very good approximation of integrals of smooth functions with compact support.

Now let $\mathbb{R}^\bullet$ denote the multiplicative group of positive real numbers. For each $y \in \mathbb{R}^\bullet$, let us consider the infinite measure, m_y, defined on smooth functions with compact support in $\mathbb{R}^\bullet$, by the formula:

$$m_y(f) = \sum_{n\in\mathbb{N}} y\varphi(n) f(y^{\frac{1}{2}}n) \tag{1}$$

where $\mathbb{N} = \{1, 2, \ldots\}$ is the set of natural numbers and $\varphi(n) = n\prod_{p|n}(1 - \frac{1}{p})$ is Euler's totient function, which counts the number of integers which are relatively prime to a given integer, and are lesser or equal to that integer. In fact, for every $r \geqq 0$, r an

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