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ESTIMATES FOR THE HYPERBOLIC METRIC

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Abstract. Bounds for the density of the hyperbolic metric of a hyperbolic region X in the complex plane $\boldsymbol{C}$ or on the Riemann sphere $\boldsymbol{P}$ are given in terms of the euclidean or spherical distance to the boundary of X. Also, bounds for the infimum of the density of the hyperbolic metric are given in terms of the supremum of the radii of all disks in X. These bounds are related to various Landau constants and are implicit in previous work on finding lower bounds for Landau constants.

1. Introduction. Let X denote a hyperbolic region in the complex plane $\boldsymbol{C}$; that is, $\boldsymbol{C}\backslash X$ contains at least two points. The hyperbolic, or Poincaré, metric on X is denoted by $\lambda_X(z)|dz|$. It is a complete Riemannian metric on X with constant curvature -4. Recall that

$$\lambda_{\boldsymbol{D}}(z)|dz|=\frac{|dz|}{1-|z|^2},$$

where $\boldsymbol{D}=\{z:|z|<1\}$ is the unit disk. Typically, there is no explicit formula for the density $\lambda_X(z)$ of the hyperbolic metric, so estimates are useful. However, there are few results that deal explicity with the size of the hyperbolic metric. Let us survey some of these. Ahlfors ([1], [2]) gave analytic bounds in case X is the thrice punctured sphere. Often, one is interested in bounds for $\lambda_X(z)$ in terms of the geometric quantity $\delta_X(z)$ which is the enclidean distance from z to the boundary of X. The upper bound $\lambda_X(z)\leqq 1/\delta_X(z)$ is a direct consequence of Schwarz' lemma [6, p. 45]. If X is simply connected, then $\lambda_X(z)\geqq 1/4\delta_X(z)$ [6, p. 45]. This lower bound is equivalent to the Koebe one-quarter theorem. If X is convex, then the factor 4 in the lower bound can be replaced by 2 [9]. Blevins [4] obtained a sharp lower bound for simply connected regions that are bounded by a quasiconformal circle. Beardon and Pommerenke ([3], [12]) investigated bounds in terms of $\delta_X(z)$ and another geometric quantity. In particular, they determined a necessary and sufficient condition on a region X for the existence of a positive constant c such that $\lambda_X(z)\geqq c/\delta_X(z)$. The condition is that there exists a positive constant M such that the modulus of any annulus in X that separates $\partial X\cup\{\infty\}$ is at most M. Hence, it is necessary that ∂X have no isolated points.

We are interested in obtaining a lower bound for $\lambda_X(z)$ in terms of $\delta_X(z)$ that is valid even if the boundary of X has isolated points. The clue to the

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