T. NAKAMURA
KODAI MATH. J.
2 (1979), 123—129

# A NOTE ON ENDOMORPHISM RINGS OF ABELIAN VARIETIES OVER FINITE FIELDS

BY TETSUO NAKAMURA

Let $p$ be a prime and let $A$ be a simple abelian variety over a finite field $k$ with $p^a$ elements. In this note we ask some sufficient conditions that the endomorphism ring of $A$ over $k$ is maximal at $p$. Our result includes the first part of theorem 5.3 in Waterhouse [5]. The related facts should be referred to [5].

**§1.** Let $\mathrm{End}_k(A)$ be the ring of $k$-endomorphisms of a simple abelian variety $A$ over a finite field $k$ with $p^a$ elements. We shall always assume that $\mathrm{End}_k(A)$ is commutative. Then there exist a $CM$ field $E$ and an isomorphism $i_A: E \to \mathrm{End}_k(A)\otimes \boldsymbol{Q}$. Let $R=\imath_A{}^{-1}(\mathrm{End}_k(A))$ and let $K$ be the totally real subfield of index 2 in $E$. Let $f_A$ be the Frobenius endomorphism of $A$ over $k$ and put $\pi=\imath_A{}^{-1}(f_A)$. Then $\pi$ is a Weil $p^a$-number, i.e. an algebraic integer such that $|\pi|^2=p^a$ in all embeddings of $E=\boldsymbol{Q}(\pi)$ into $\boldsymbol{C}$. Let $w$ be a place of $K$ above $p$ and $v$ be a place of $E$ with $v|w$. Then we have the following three cases;

(1) $v(\pi)=0$ or $v(\pi)=v(p^a)$.

(2) $v(\pi)=v(p^a\pi^{-1})$.

(3) $v(\pi)\neq v(p^a\pi^{-1})$ and $0<v(\pi)<v(p^a)$.

We call that $w$ is of type (1) (resp., (2), (3)) if $v$ satisfies (1) (resp., (2), (3)). This is independent of the choice of $v$ with $v|w$. Let $K_w$ be the completion of $K$ at $w$ and let

$$\begin{aligned} G_w &=(G_{1,0})^{[K_w:\boldsymbol{Q}_p]}, \quad \text{if } w \text{ is of type (1)}, \\ &=(G_{1,1})^{[K_w:\boldsymbol{Q}_p]}, \quad \text{if } w \text{ is of type (2)}, \\ &=G_{s,t}+G_{t,s}, \quad \text{if } w \text{ is of type (3)}, \end{aligned}$$

where $s=s(w)=[K_w:\boldsymbol{Q}_p]v(\pi)/v(p^a)$ and $t=t(w)=[K_w:\boldsymbol{Q}_p]v'(\pi)/v'(p^a)$ with the other place $v'$ of $E$ above $w$. Then the formal group $\hat{A}$ of $A$ is isogenous to $\sum_{w|p} G_w$(over the algebraic closure of $k$.) (cf. Manin [1], Chap. IV).

Now let $T_pA$ be the Dieudonné module of $\hat{A}$. Let $W=W(k)$ be the ring of Witt vectors over $k$ and $\sigma$ the automorphism of $W$ induced by the Frobenius

Received October 13, 1977.