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AUTOMORPHISMS OF THE GALOIS GROUP OF THE ALGEBRAIC CLOSURE OF THE RATIONAL NUMBER FIELD

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For any Galois extension K/k, let $\mathrm{Gal}(K/k)$ be the topological Galois group of K/k. Let Q be the rational number field and let $\bar{Q}$ be the algebraic closure of Q. For algebraic number field K, let $\tilde{K}$ be the composite of all solvable extensions of K, and put $G_K=\mathrm{Gal}(\bar{Q}/K)$ and $\tilde{G}_K=\mathrm{Gal}(\tilde{K}/K)$.

In [1] and [2] Neukirch proved that for algebraic number fields K_1 and K_2 which are finite Galois extensions of Q, $G_{K_1}\simeq G_{K_2}$ (or $\tilde{G}_{K_1}\simeq\tilde{G}_{K_2}$) implies $K_1=K_2$, and in [2] he gave a conjecture to the effect that any automorphism of G_Q (or $\tilde{G}_Q$) is inner. By his theorem we have that $\sigma(G_K)=G_K$, for any automorphism σ of G_Q (or $\tilde{G}_Q$) and for any number field K which is a finite Galois (or solvable, res.) extension of Q; thus we have that σ induces an automorphism σ_K of $\mathrm{Gal}(K/Q)$. If by $\mathrm{Aut}_0(\mathrm{Gal}(K/Q))$ we denote the subgroup of the automorphism group $\mathrm{Aut}(\mathrm{Gal}(K/Q))$ of $\mathrm{Gal}(K/Q)$, consisting of those elements which leave any normal subgroup of $\mathrm{Gal}(K/Q)$ invariant, we have that the mapping $\sigma\mapsto(\sigma_K)_K$ gives a canonical isomorphism of the automorphism group $\mathrm{Aut}(G_Q)$ (or $\mathrm{Aut}(\tilde{G}_Q)$) onto the projective limit $\varprojlim \mathrm{Aut}_0(\mathrm{Gal}(K/Q))$, where K runs among the number fields which are finite Galois (or solvable, resp.) extensions of Q. It is shown that the above conjecture is true if and only if any $\sigma\in\mathrm{Aut}(G_Q)$ (or $\mathrm{Aut}(\tilde{G}_Q)$) induces an inner automorphism σ_K for any finite Galois (or solvable, resp.) extension K of Q.

As Neukirch pointed out in [2], it is natural to consider some kind of group extensions to solve this problem. In this note we shall show that σ_K is inner for a certain class of finite Galois (or solvable) extensions K of Q, at least for any finite abelian extension K of Q.

Let $G=\{g, g_1, g_2, \cdots\}$ be a finite group and let $A=\{a, a_1, a_2, \cdots\}$ be a finite abelian group. Let θ be a homomorphism of G into the automorphism group(A) of A and let

$$G\times A\ni(g, a)\longmapsto g\circ a=\theta(g)(a)\in A$$

be the operation of G on A by θ. Let $\hat{G}$ be the semidirect product $A\times_\theta G$ of A and G by θ: i.e. $\hat{G}$ is the group which is $A\times G$ as set and in which the group operation is given by

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