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ON A METRIC INDUCED BY ANALYTIC CAPACITY

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Dedicated to Professor Yukinari Tôki on his 60th birthday

1. In our previous paper [6] we gave a conjecture that the metrics $c_\beta(z)|dz|$ and $\sqrt{\pi\tilde{K}(z, z)}|dz|$ have negative curvatures $\leqq -4$; here $c_\beta(z)$ and $\tilde{K}(z, z)$ are the capacity and the Bergman kernel of exact analytic differentials on an open (non-trivial) Riemann surface. In the present paper we shall show that the curvature of the metric $c_B(z)|dz| \leqq -4$ for plane regions $\Omega \notin O_{AB}$, where $c_B(z)$ denotes the analytic capacity of Ω at z. In order to verify $c_B(z) \in C^2$ we prove that $c_B(z)$ is real analytic. This enables us to answer a question of Havinson [4], namely "Does the sequence of extremal functions ψ_n in the dual problem of Schwarz's lemma in Ω_n converge as $\{\Omega_n\}$ exhausts Ω?".

2. Let Ω be a plane region $\notin O_{AB}$. The analytic capacity $c_B(\zeta)$ is given by $\sup |f'(\zeta)|$ in the family of analytic functions satisfying $f(\zeta)=0$ and $|f(z)| \leqq 1$. Let $\{\Omega_n\}$ be a canonical exhaustion of Ω such that the boundary of Ω_n consists of a finite number of analytic curves. Let $c_n(\zeta)$ be the analytic capacity of Ω_n. Then $\{c_n(\zeta)\}$ is decreasing and tends to $c_B(\zeta)$. There exist extremal functions f_n such that $f'_n(\zeta)=c_n(\zeta)$ and $f'_0(\zeta)=c_B(\zeta)$. It is known that those extremal functions are unique [4]. The function $\log c_B(\zeta)$ is subharmonic [2].

In every Ω_n there exists the Szegö kernel $k_n(z, \zeta)$ and its adjoint kernel $l_n(z, \zeta)$ [3]. $k_n(z, \zeta)$ is hermitian and analytic with respect to z and $\bar{\zeta}$. Further the following facts are known [3]:

$$f_n(z) = \frac{k_n(z, \zeta)}{l_n(z, \zeta)} \quad \text{with} \quad c_n(\zeta) = 2\pi k_n(\zeta, \zeta) \tag{1}$$

and

$$|k_n(z, \zeta)|^2 \leqq |k_n(z, z)||k_n(\zeta, \zeta)|.$$

Thus $k_n(z, \zeta)$ is uniformly bounded on every compact subset and hence forms a normal family of analytic functions of two variables $z, \bar{\zeta}$. We will show that $\{k_n(z, \zeta)\}$ converges to a function $k(z, \zeta)$ uniformly on every compact subset of Ω.

Suppose that there exist two limit functions $k(z, \zeta)$ and $k^*(z, \zeta)$. We may assume $0 \in \Omega$. The difference $k(z, \zeta) - k^*(z, \zeta)$ has an expansion in a polydisc $\{|z|<r\} \times \{|\zeta|<r\}$

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