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A PROOF OF THE BIEBERBACH CONJECTURE FOR THE FOURTH COEFFICIENT

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1. Introduction. Let $f(z)$ be a normalized regular function univalent in the unit circle $|z|<1$

$$f(z)=z+\sum_{n=2}^{\infty} a_n z^n.$$

So far as the present author knows, up to the present time, there have appeared five proofs of $|a_4|\leqq 4$; [2], [3], [4], [5], [7]. In this paper we shall give another proof of $|a_4|\leqq 4$.

THEOREM 1.

$$\Re\{a_4-2a_2a_3+a_2^3+2\alpha(a_3-a_2^2)+(1+\alpha)^2a_2\}\leqq 2(1+\alpha+\alpha^2)$$

for $\alpha\geqq 0$.

We shall give here two proofs of this theorem. One is due to Schiffer's variational method together with Bombieri's recent result [1] and the other is due to Grunsky's inequality [6]. Then we shall prove

THEOREM 2. $|a_4|\leqq 4$. *Equality occurs only for* $z/(1-e^{i\varepsilon}z)^2$, ε: *real.*

It is well known that Grunsky's inequality gives a quite easy proof of $|a_4|\leqq 4$ [2]. Hence our proof should be considered as a non-elementary proof from a methodological point of view. Our emphasis lies in the form of the corresponding Schiffer differential equation, which does not have any perfect square form.

2. Proof of Theorem 1. Let us consider the problem

$$\max \Re\{a_4-2a_2a_3+a_2^3+2\alpha(a_3-a_2^2)+(1+\alpha)^2a_2\}.$$

Then the image of $|z|=1$ by any extremal functions satisfies

$$\left(\frac{dw}{dt}\right)^2\frac{1}{w^5}[(1+\alpha)^2w^2+(a_2+2\alpha)w+1]+1=0$$

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