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ON PRIME ENTIRE FUNCTIONS

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§1. An entire function $F(z)=f\circ g(z)$ is said to be prime if every factorization of the above form implies that one of the functions $f(z)$ or $g(z)$ is linear.

Ozawa [5] has recently proved the following.

THEOREM A. *Let $F(z)$ be an entire function of order ρ, $1/2<\rho<1$ and with only negative zeros. Assume that $n(r)\sim\lambda r^{\rho}$, $\lambda>0$ where $n(r)$ indicates the number of zeros of $F(z)$ in $|z|<r$. Further assume that there are two indices j and k such that a_j, a_k are zeros of $F(z)$ whose multiplicities p_j, p_k satisfy $(p_j, p_k)=1$. Then $F(z)$ is prime.*

The purpose of this note is to extend Theorem A to higher orders and to prove the following.

THEOREM. *Let $F(z)$ be an entire function of non-integral order ρ $(>1/2)$, with only negative zeros. Assume that $n(r)\sim\lambda r^{\rho}$, $\lambda>0$. Further assume that there are two indices j and k such that a_j, a_k are zeros of $F(z)$ whose multiplicities p_j, p_k satisfy $(p_j, p_k)=1$. Then $F(z)$ is prime.*

In order to prove this we quote several known results.

LEMMA 1. (Edrei [1]). *Let $f(z)$ be an entire function. Assume that there exists an unbounded sequence $\{h_\nu\}_{\nu=1}^{\infty}$ such that all the roots of the equations $f(z)=h_\nu$, $\nu=1, 2, \cdots$, be real. Then $f(z)$ is a polynomial of degree at most two.*

LEMMA 2. (Pólya [6]). *Suppose that $f(z)$, $g(z)$ are entire functions and that $\phi(z)=f\circ g(z)$ is of finite order. Then either $g(z)$ is a polynomial or $f(z)$ is of order zero.*

LEMMA 3. (Hardy-Littlewood [2]). *If $F(z)$ is a positive integrable function such that, when $t\to 0$,*

$$\int_0^{\infty} F(x)e^{-xt}dx\sim t^{-\beta} \qquad (\beta>0),$$

then, when $x\to\infty$,

$$\int_0^{x} F(u)du\sim\frac{x^{\beta}}{\Gamma(\beta+1)}.$$

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