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ON THE UNIQUENESS OF THE EXTREMAL FUNCTION OF HARMONIC LENGTH PROBLEM AND ITS APPLICATION

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1. Landau and Osserman [4] introduced the notion of harmonic length as follows: Let R be an arbitrary Riemann surface, we denote by U_R the family of all functions u harmonic on R satisfying $0<u<1$ on R. Let γ be an arbitrary cycle on R. We call

$$h_R(\gamma)=\sup_{u\in U_R}\int_\gamma *du$$

the harmonic length of γ.

They showed the following

THEOREM A. *Let D be a Dirichlet region on a Riemann surface R and let u be a harmonic measure in D. If γ is homologous in D to a level locus of u, then u is the unique extremal function in determining $h_D(\gamma)$.*

Here a Dirichlet region means a relatively compact region on a Riemann surface whose boundary is regular for the Dirichlet problem, and which has at least two boundary components.

They applied this theorem to problems of conformal rigidity of plane Dirichlet regions.

Recently, Suita and the author [7] have shown the following improvement of theorem A. An English reference will be made to Suita [6].

THEOREM B. *Let R be an arbitrary Riemann surface and let γ be a dividing cycle relative to a regular partition (A, B) on R. If $h_R(\gamma)>0$, the function satisfying*

$$h_R(\gamma)=\int_\gamma *du_0, \qquad u_0\in U_R$$

is unique and coincides with the harmonic measure of B.

In the present paper we shall show the uniqueness of the extremal function in determining the harmonic length of any cycle on a finite Riemann surface. Using the uniqueness, we shall give an elementary proof of a theorem of Huber in the case of finite Riemann surfaces [3]. The author expresses his heartiest thanks to Professor N. Suita for his kind suggestion in preparing this paper.

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