ON PRIME ENTIRE FUNCTIONS

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§ 1. An entire function $F(z) = f \circ g(z)$ is said to be prime (pseudo-prime) if every factorization of the above form implies that one of the functions $f(z)$ or $g(z)$ is linear (a polynomial). It is almost trivial that $z^p$ with a prime $p$ is prime. However it is hard to say that known examples of prime transcendental entire functions are rich in number. It is known that $e^z + z$ is prime, which had been stated in Rosenblum's pioneering paper [7] without proof and was explicitly proved by Gross [3]. For the pseudo-primeness we had published several papers [5]. Our methods may be classified two types. One depends upon the Picard theorem and the other the following elegant theorem due to Edrei [2]:

**Lemma.** Let $f(z)$ be an entire function. Assume that there exists an unbounded sequence $\{h_\nu\}_{\nu=1}^\infty$ such that all the roots of the equations $f(z) = h_\nu$, $\nu = 1, 2, \ldots$ lie on the real positive axis. Then $f(z)$ is prime.

However we need another consideration in order to assure the primeness of individual functions.

In this paper we shall give a method, which guarantees the primeness. This method has close connection with the famous Wiman theorem. In the last part we shall give several examples of prime functions, whose proof depends upon their special forms.

§ 2. We shall prove the following criterion of primeness.

**Theorem 1.** Let $F(z)$ be an entire function of order less than one:

$$
\prod_{l=1}^\infty \left(1 - \frac{z}{a_l}\right)^{p_l}, \quad a_l > 0, \quad a_{l+1} > a_l.
$$

Suppose that there are two indices $j$ and $k$ such that $(p_j, p_k) = 1$. Further suppose that there is a sequence $\{r_\nu\}$ such that $a_{n-1} < r_\nu < a_n$ and $\lim_{n \to \infty} F(r_\nu) = \infty$. Then $F(z)$ is prime.

**Proof.** Let $F(z)$ be $f \circ g(z)$. Assume that $f(w)$ is transcendental. Then its order must be less than or equal to the order of $F(z)$. Hence $f(w) = 0$ has an infinite number of roots $\{w_\nu\}$, $w_\nu \to \infty$ as $n \to \infty$. Consider the equations $g(z) = w_\nu$, $n = 1, 2, \ldots$. All their roots lie on the real positive axis. Then by Edrei's theorem

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