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A CONVEXITY IN METRIC SPACE AND NONEXPANSIVE MAPPINGS, I

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1. Introduction.

In this paper, we shall discuss convexity and fixed point theorems in certain metric space which are described in an abstract form. At first we shall introduce a concept of convexity in a metric space and study the properties of the space which we call a convex metric space. Futhermore, we formulate some fixed point theorems for nonexpansive mappings (i.e. mappings which do not increase distances) in the space. Consequently, these generalize fixed point theorems which have been previously proved by Browder [1], Kirk [6] and the author [7] in a Banach space.

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2. Definitions and propositions.

Throughout this paper, we consider a metric space X with a *convex structure* such that there exists a mapping W from $X\times X\times[0,1]$ to X (i.e. $W(x, y; \lambda)$ defined for all pairs $x, y\in X$ and λ $(0\leqq\lambda\leqq 1)$) and valued in X satisfying

$$(*)\qquad d(u, W(x, y; \lambda))\leqq\lambda d(u, x)+(1-\lambda)d(u, y)$$

for all $u\in X$ and call this space X a *convex metric space*. A Banach space and each of its convex subsets are, of course, convex metric spaces. But a Fréchet space is not necessary a convex metric space. There are many examples of convex metric spaces which are not imbedded in any Banach space. We give two preliminary examples here.

EXAMPLE 1. Let I be the unit interval $[0,1]$ and X be the family of closed intervals $[a_i, b_i]$ such that $0\leqq a_i\leqq b_i\leqq 1$. For $I_i=[a_i, b_i]$, $I_j=[a_j, b_j]$ and λ $(0\leqq\lambda\leqq 1)$, we define a mapping W by $W(I_i, I_j; \lambda)=[\lambda a_i+(1-\lambda)a_j,\ \lambda b_i+(1-\lambda)b_j]$ and define a metric d in X by the Hausdorff distance, i.e.

$$d(I_i, I_j)=\sup_{a\in I}\{|\inf_{b\in I_i}\{|a-b|\}-\inf_{c\in I_j}\{|a-c|\}|\}.$$

EXAMPLE 2. We consider a linear space L which is also a metric space with

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