KŌDAI MATH. SEM. REP.
20 (1968), 305—313

ON THE SOLUTION OF THE FUNCTIONAL EQUATION $f\circ g(z)=F(z)$, V

BY MITSURU OZAWA

In our previous paper we discussed the transcendental unsolvability of the functional equation $f\circ g(z)=F(z)$. In this note we shall extend some results in [4] to a more general class of functions and make use of the same terminology "transcendental solvability". Our basic tool is an elegant theorem of Edrei-Fuchs [2].

THEOREM 1. *Let $f(z)$ be an entire function of the form $P(z)e^{M(z)}$ with a polynomial $P(z)$. Assume that there exist two constants a, b such that $|a|\neq|b|$, $ab\neq 0$ and that $f(z)=a$ and $f(z)=b$ have their solutions on p straight lines $l_1, \cdots, l_p$, almost all, any two of which are not parallel with each other. Then $f(z)$ reduces to a polynomial.*

Proof. By Edrei-Fuchs' theorem in [2] $f(z)$ must be of finite order and hence $M(z)$ must be a polynomial. Denote it by

$$\alpha_n z^n+\alpha_{n-1}z^{n-1}+\cdots+\alpha_1 z+\alpha_0, \qquad \alpha_n\neq 0.$$

By a suitable change of variable we have

$$M(z)=z^n+\alpha_{n-2}z^{n-2}+\cdots+\alpha_1 z+\alpha_0$$

with new α_j. Hence our problem reduces to solve the following equation

$$(A_m z^m+\cdots+A_0)\exp(z^n+\alpha_{n-2}z^{n-2}+\cdots+\alpha_0)=a.$$

We have asymptotically

$$z^n\left(1+O\left(\frac{1}{z^2}\right)\right)=\log\frac{a}{A_m e^{\alpha_0}}+2q\pi i.$$

Hence the given p straight lines $l_1, \cdots, l_p$ must be parallel to one of

$$\arg z=\pm\frac{\pi}{2n}+\frac{2s}{n}\pi, \qquad s=0, \cdots, n-1,$$

respectively. Assume that l_1 is parallel to a radius given by

$$Re^{\pi i/2n}.$$

Then l_1 can be represented as $x_0+R\exp(i\pi/2n)$ with a real x_0. Let

Received November 20, 1967.