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INVARIANT SUBFIELDS OF RATIONAL FUNCTION FIELDS

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Let K be the rational function field $k(X_1, X_2, \cdots, X_n)$ of variables $X_1, X_2, \cdots, X_n$ over a field k. Let M be the vector space $\sum_{i=1}^{n} k\cdot X_i$ over k. Let $\mathfrak{g}$ be a finite group operating on K, induced by a representation ρ of $\mathfrak{g}$ with representation space M. Let L be the subfield of K consisting of elements which are invariant under $\mathfrak{g}$. The problem to consider here is whether L is purely transcendental over k. This problem has been answered affirmatively in the following cases: (0) $\mathfrak{g}$ is the symmetric group permuting $X_1, X_2, \cdots, X_n$, (1) $\mathfrak{g}$ is abelian and k is the complex number field, (2) $\mathfrak{g}$ is a cyclic group of order n, ρ is its regular representation and k contains the primitive n-th roots of unity, provided that the characteristic of k does not divide n (cf. [5]) and (3) k is of characteristic $p>0$, $\mathfrak{g}$ is a p-group and ρ is its regular representation (cf. [2], [3] and [4]). In this note we shall give a principle, written in language of algebraic groups, which covers the three cases (1), (2) and (3), and which may be applied to other cases where $\mathfrak{g}$ is soluble.

A connected algebraic group G is called k-soluble if there exists a normal chain $G_0=G\supset G_1\supset G_2\supset\cdots\supset G_r=\{e\}$ such that G_i is defined over k and G_i/G_{i+1} is isomorphic to G_a or G_m over k, where G_a and G_m are the additive group of the universal domain Ω and the multiplicative group of non-zero elements of Ω. The following property of k-soluble algebraic groups is used here (cf. [6]): let G be a k-soluble algebraic group; let V be a homogeneous space with respect to G over k, then the function field $k(V)$ over k is purely transcendental over k.

From this we have

(P) Let G be a k-soluble algebraic group such that $k(G)=K$; let $\mathfrak{g}$ be a finite subgroup of G which is rational over k such that the invariant subfield of K by the left translations of $\mathfrak{g}$ is L, then L is purely transcendental over k.

In fact, there exists the quotient variety $G/\mathfrak{g}$, defined over k, which is a homogeneous space with respect to G over k.

Let us consider the case where $\mathfrak{g}$ is abelian.

LEMMA. *Let $\mathfrak{g}$ be a finite abelian subgroup of $GL(n, k)$ of exponent m. Then, if k contains the primitive m-th roots of unity, there exists $x\in GL(n, k)$ such that $x\cdot\mathfrak{g}\cdot x^{-1}$ is contained in the set of matrices of the form*

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