ON THE AUTOMORPHISM RING OF DIVISION ALGEBRAS

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1. Introduction.

Let A be an (associative) ring with an identity 1 and S a subring of A containing 1. Suppose S is Galois in A in the sense that $I(H(S))=S$, where $H(S)$ is the group of all automorphisms of A leaving S elementwise invariant (i.e. the Galois group of A over S and $I(H(S))$ is the set of all elements of A invariant under every automorphism of $H(S)$.[1] The Galois group $\mathfrak{G}=H(S)$ and the set S_R of right multiplications by elements of S generate a subring $\mathfrak{R}=\mathfrak{G}S_R=S_R\mathfrak{G}$ of the ring $\mathfrak{E}$ of S-endomorphisms of A as an S-left module. The ring $\mathfrak{R}$ is called the *automorphism ring* of A over S.

In a series of papers [7—9], Kasch investigated the properties of $\mathfrak{R}$ and of A as an $\mathfrak{R}$-module, assuming mostly that A is a simple ring satisfying minimum condition for right ideals (a division ring, in particular) and that S is a Galois subring of A such that $[A: S]<\infty$.[2] The main problem he discussed was: Under what conditions $\mathfrak{R}$ and A are isomorphic as $\mathfrak{R}$-modules? The problem is related to the normal basis theorem and to this he gave a quite satisfactory answer ([7]).[3] Also, he started the study of the structure of $\mathfrak{R}$ and of A as an $\mathfrak{R}$-module.[4] In this direction, he obtained the following remarkable result ([9]).

Let $A=Z_m$ be the total matrix algebra over a commutative field Z of degree $m>1$ and $\mathfrak{G}$ the group of all inner automorphisms of A (i.e. the Galois group of A over Z). Suppose that Z is not the prime field of characteristic 2 and that the degree m is not divisible by the characteristic of Z. If $\mathfrak{R}=\mathfrak{G}Z_R=\mathfrak{G}Z$ is the automorphism ring of A over Z then:

(a) A is completely reducible as $\mathfrak{R}$-module and has a (unique) direct sum decomposition $A=Z\oplus B$, where $B=[A, A]$ is the submodule of A generated by (additive) commutators $[a_1, a_2]=a_1a_2-a_2a_1$, $a_1, a_2\in A$.

(b) $\mathfrak{R}$ induces all linear transformations of B over Z.

(c) $\mathfrak{R}$ is semi-simple and moreover is expressible as the direct sum of Z and Z_{m^2-1}, the total matrix algebra of degree m^2-1 over Z; hence $[\mathfrak{R}: Z]=(m^2-1)^2+1$.

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1) Cf. Jacobson [5], Chapters 6–7.

2) In the case of simple A, we have to add some other conditions to the definition of Galois subrings. (The definition that we mentioned above is, in this case, too general.)

3) A supplementary result was obtained by Nagahara-Onodera-Tominaga [10].

4) Concerning this problem, only preliminary results have been obtained.