ON MARKOV CHAINS WITH REWARDS

By Hirohisa Hatori

1. Let $X_0, X_1, X_2, \cdots$ be a Markov chain with the state space $\boldsymbol{S}=\{1, 2, \cdots, N\}$ and the transition probability matrix $P=(p_{ij})$, which earns r_{ij} dollars when it makes a transition from state i to state j. We call r_{ij} the "reward" associated with the transition from i to j. Let us define $v_i(n)$ as the expectation of the total earnings $R(n)$ in the next n transition if the system is now in state i. Howard [1] has given the recurrence relation

$$(1) \qquad v_i(n)=\sum_{j=1}^{N} p_{ij}r_{ij}+\sum_{j=1}^{N} p_{ij}v_j(n-1) \qquad (i=1, 2, \cdots, N;\ n=1, 2, \cdots)$$

and the asymptotic form that $v_i(n)$ assumes for large n. For practical applications, it is desirable to find the variance and the asymptotic behavior of $R(n)$. And it is also intersting from the theorical view-point. In this paper, we shall try to find the asymptotic form that $\mathrm{Var}\,(R(n))$ assumes for large n and to give a proof of the limit theorem for $R(n)$ as $n\to\infty$, which is essentially equivalent to the limit theorem for finite regular Markov chains.

2. Since, by the definition of Markov chains, we have

$$\begin{aligned} &P\{X_1=i_1, X_2=i_2, \cdots, X_n=i_n \mid X_0=i\}\\ &=P\{X_1=i_1 \mid X_0=i\}P\{X_2=i_2 \mid X_1=i_1, X_0=i\}\cdots P\{X_n=i_n \mid X_{n-1}=i_{n-1}, \cdots, X_0=i\}\\ &=p_{ii_1}p_{i_1i_2}\cdots p_{i_{n-1}i_n},\end{aligned}$$

the moment generating function of the probability distribution of $R(n)$ under the condition $X_0=i$ is equal to

$$(2) \qquad \begin{aligned} \phi_{in}(\theta)&\overset{\mathrm{def}}{=}E\{e^{\theta R(n)} \mid X_0=i\}\\ &=\sum_{i_1,\cdots,i_n} e^{\theta(r_{ii_1}+r_{i_1i_2}+\cdots+r_{i_{n-1}i_n})}P\{X_1=i_1, \cdots, X_n=i_n \mid X_0=i\}\\ &=\sum_{i_n}\sum_{i_1,\cdots,i_{n-1}} e^{\theta r_{ii_1}}p_{ii_1}e^{\theta r_{i_1i_2}}p_{i_1i_2}\cdots e^{\theta r_{i_{n-1}i_n}}p_{i_{n-1}i_n}.\end{aligned}$$

Introducing the $N\times N$ matrix $\Pi(\theta)\overset{\mathrm{def}}{=}(e^{\theta r_{ij}}p_{ij})$ with elements $e^{\theta r_{ij}}p_{ij}$ and two N-dimentional vectors

$$\boldsymbol{\phi}_n(\theta)=\begin{bmatrix}\phi_{1n}(\theta)\\ \phi_{2n}(\theta)\\ \vdots\\ \phi_{Nn}(\theta)\end{bmatrix} \quad \text{and} \quad \mathbf{e}=\begin{bmatrix}1\\ 1\\ \vdots\\ 1\end{bmatrix},$$

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