MINIMAL SLIT REGIONS AND LINEAR OPERATOR METHOD

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1. Let Ω be a plane region containing the point at infinity. Let $\mathfrak{F}_\Omega$ be the family of all the univalent functions f on Ω having the expansion

$$(1) \qquad f(z)=z+\frac{c}{z}+\cdots$$

about ∞. The function maximizing (minimizing) $\operatorname{Re} c$ in $\mathfrak{F}_\Omega$ exists and is determined uniquely, which we denote by $\varphi_\Omega(\psi_\Omega$, resp.).

The image region $\psi_\Omega(\Omega)$ $(\psi_\Omega(\Omega))$ is a horizontal (vertical) parallel slit plane. Conversely, however, an arbitrary horizontal (vertical) parallel slit plane can not be, in general, the image of an Ω under $\varphi_\Omega(\psi_\Omega)$; in fact the measure of $\varphi_\Omega(\Omega)^c$ and $\psi_\Omega(\Omega)^c$ vanish. Accordingly, with Koebe, we introduce the following:

Definition. A horizontal (vertical) parallel slit plane Δ is said to be *minimal* if $\Delta=\varphi_\Omega(\Omega)$ $(\Delta=\psi_\Omega(\Omega)$, resp.) for an Ω containing ∞.

The minimality of slit regions is characterized by moduli of quadrilaterals (Grötzsch [2]) or extremal length (Jenkins [3]). From the point of view of the latter a number of interesting properties are derived in Suita's paper in these Reports [8].

The linear operator method due to Sario [6] (see also Chapter III of the book by Ahlfors-Sario [1]) gives us another approach to φ_Ω and ψ_Ω. From this a characterization of minimality is derived, which is rather similar to the original one due to Koebe [4]. It is the purpose of the present paper to show how to use this method to prove alternatively a part of Suita's results mentioned above.

2. We begin with reviewing the definition of the normal linear operators L_0 and L_1 in Ahlfors-Sario [1].

Let W be an open Riemann surface, let V be a regularly imbedded non-compact subregion with compact relative boundary α. For any real analytic function f on α, consider the problem of constructing the function u such that

$$(2) \qquad \text{harmonic on } V\cup\alpha, \qquad u=f \text{ on } \alpha.$$

If V is the interior of a compact bordered surface we can assign the behavior of u on $\beta=$(border of $V)-\alpha$ so that u may be determined uniquely. For our purpose the following two are necessary:

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