ON RIEMANN-LIOUVILLE INTEGRAL OF ULTRA-HYPERBOLIC TYPE

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1. Introduction.

Riesz has persuited the many types, elliptic and hyperbolic types, of Riemann-Liouville integral since 1933. Now let r_{PQ} be the distance between two points P and Q, then we call the following integral the Riemann-Liouville integral

$$I^{\alpha}f(P)=\frac{1}{H_m(\alpha)}\int f(Q)r_{PQ}^{\alpha-m}\,dQ.$$

Here the range of integration is taken suitably according to the above mentioned types. Further $H_m(\alpha)$ corresponds to the gamma function in the old theory of Riemann-Liouville integral and it depends on the dimension m of the space and on the constant α. About this there hold the fundamental formulas

$$I^{\alpha}(I^{\beta}f(P))=I^{\alpha+\beta}f(P),\ \Delta I^{\alpha+2}f(P)=\pm I^{\alpha}f(P)\quad\text{and}\quad I^{0}(P)=f(P).$$

Using the Riemann-Liouville integral, Riesz [6] established the general potential theory in m-dimensional Euclidean space. Further Frostman [3] proved elegantly the fundamental theorem on the equilibrium potential in his α-dimensional potential theory. In his proof a lemma played its essential part. This lemma can be obtained from the theory of the Riemann-Liouville integral and it is

$$\int_{\Omega_3}\frac{1}{r_{PM}^{k}}\frac{1}{r_{MQ}^{l}}\,dM=H_m(k,l)\frac{1}{r_{PQ}^{k+l-3}}.$$

In addition, this equality has many applications to other branches of analysis; cf. Nozaki [5].

Next let the distance r_{PQ} of two points $P(x)$ and $Q(\xi)$ be

$$r_{PQ}{}^2=(x_1-\xi_1)^2-(x_2-\xi_2)^2-\cdots-(x_m-\xi_m)^2.$$

Then Riesz called the space with this distance r_{PQ} (hyperbolic distance) the Lorentzian space. In this space he constructed the theory of Riemann-Liouville integral. Using this integral he solved Cauchy problem which is one of the branches of the theory of the hyperbolic partial differential equations. Riesz's theories were given in his splended paper [7].

Now in the present paper, the author will intend to extend Riesz's results more

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