LINEAR MAPPINGS AMONG THE FUNCTION CLASSES ON RIEMANN SURFACES

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Introduction

Let W be an open Riemann surface and G a non-compact subregion of W, whose relative boundary ∂G consists of at most a countably infinite number of analytic components which do not cluster at any compact part of W. Let R be a collection of compact or non-compact subregions of G all boundary components of which consists of a finite number of analytic curves and cluster at only the ideal boundary of W. We assume further that any point on the relative boundary is freely accessible from both sides of curves.

Let $\{W_n\}$ be a sequence of compact domains satisfying the following conditions:
(1) $W_n \subset W_{n+1}$,
(2) ∂W_n consists of a finite number of components each of which consists of a finite number of analytic curves, and
(3) for any compact subset K there exists such a number $n_0(K)$ that K is contained in W_n for any $n \geqq n_0(K)$.
We say this sequence $\{W_n\}$ an exhaustion of W, however our notion of exhaustion presented here is different from that usually availed. Indeed there may exist a finite number of islands any member of which does not separate any ideal boundary element and a stronger condition $\overline{W}_n \subset W_{n+1}$ than the one claimed in (1) is not postulated here.

We shall make use of the following notations:

H: a class of harmonic functions, which is equal to zero on the relative boundary if it exists. P: a class of positive functions. B: a class of bounded functions. $A(X)$: some function class A defined in X.

We shall introduce several linear mappings among several families of functions some of which are well known.

For each $u \in PH(G)$ the upper envelope of the set of members of $PH(R)$ which are dominated by u in R is itself a member of $PH(R)$. We define a mapping T of $PH(G)$ into $PH(R)$ by requiring that for each $u \in PH(G)$ Tu shall be the above upper envelope. In another direction, we define for each $U \in PH(R)$, its standard subharmonic extension U^* by $U^* = U$ in R, $=0$ in $G-R$. Let Q_R denote the subset of $PH(R)$ consisting of those U for which U^* admits a harmonic majorant on G. We define the map S from Q_R into $PH(G)$ by requiring that SU shall be the least harmonic majorant of U^* on G. It is immediate that both T and S are positively linear. Then we see that S is univalent and $TSU = U$ for any $U \in Q_R$. If $v \in PH(G)$ is dominated by some member of $S(Q_R)$, then $v \in S(Q_R)$. Now we can define a mapping

Received July 5, 1962.