CONDITIONAL EXPECTATION IN AN OPERATOR ALGEBRA, IV (ENTROPY AND INFORMATION)

By Hisaharu Umegaki

1. Introduction.

The theory of information, created by Shannon [23], is developed by Feinstein, Kullback, MacMillan, Wiener and other American statisticians (e. g., cf. [10]), and also advanced into the ergodic theory by Gelfand, Khinchin, Kolmogorov, Yaglom and other Russian probabilists (e. g., cf. [8]). Through recent years, the theory is regarded as a new chapter in the theory of probability.

Recently, Segal [22] gave a mathematical formulation of the entropy of state of a von Neumann algebra, which contains both the cases for the theory of information and the theory of quantum statistics. Segal's theorem was reformulated in operator algebraic form by Nakamura and Umegaki [16] and independently by Davis [3].

Since the summer in 1954, Nakamura and Umegaki have investigated the concept of the conditional expectation in von Neumann algebra as a non-commutative extension of probability theory (cf. for example [13~18] and [25~28]), and in the most recent paper [18] it was applied to the theory of measurements of quantum statistics which is regarded as a non-commutative case of the theory of entropy and information. Furthermore, it may be very intersesting to develope the theory of information under functional-analysistic and operator-theoretic methods. From these points of views, we shall discuss the measure of information of integrable operators or of normal states of a von Neumann algebra. Davis [3] has independently studied on the almost same theme with Nakamura-Umegaki [16] and [18], in which he developed the theory of entropy and he simplified the proof of the theorem relative to the operator-entropy.

Now, we shall give the basic notations and describe the fundamental concepts in a von Neumann algebra which will be used throughout the present paper.

Let A be a von Neumann algebra, that is, A is a weakly closed self-adjoint algebra of bounded operators acting over a complex Hilbert space H, which contains the identity operator I. A linear functional ρ of A is said to be *positive* if $\rho(aa^*)\geqq 0$ for every $a \varepsilon A$. Such ρ is said to be *state* if $\rho(I)=1$, to be *normal* in the terminology of Dixmier [4] if $\rho(a_\alpha)\uparrow\rho(a)$ for $a_\alpha\uparrow a$, and to be *trace* if $\rho(ab)=\rho(ba)$ for every pair $a, b \varepsilon A$. The normality of state is equivalent to the complete additivity: $\sum\rho(p_\alpha)=\rho(\sum p_\alpha)$ for any disjoint family of projections $\{p_\alpha\}\subset A$ (cf. Dixmier [4]).

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