A DISTORTION THEOREM OF UNIVALENT FUNCTIONS RELATED TO SYMMETRIC THREE POINTS

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1. Let Σ be a family of functions $g(z)$ meromorphic and univalent for $|z|>1$ with Laurent expansion for $|z|>1$ given by

$$g(z)=z+c_0+\frac{c_1}{z}+\cdots.$$

The distortion inequality

$$\frac{(1-r^{-2})^2}{4r^2(1+r^{-2})^2}\leqq\frac{|g'(z)g'(-z)|}{|g(z)-g(-z)|^2}\leqq\frac{(1+r^{-2})^2}{4r^2(1-r^{-2})^2}\qquad(z=re^{i\theta})$$

for $g(z)$ belonging to Σ is easily obtained by combining the classical results. It can be also shown that the left and right equalities are attained by the functions $z+e^{i2\theta}z^{-1}$ and $z-e^{i2\theta}z^{-1}$ respectively.

We are concerned in the present paper with an analogous problem relating to symmetric three points z, $ze^{i2\pi/3}$ and $ze^{i4\pi/3}$. Analogous bounds will be obtained and the extremal functions will be closely connected with the above two functions. We remark that a known coefficient inequality $|c_2|\leqq 2/3$ can be proved from our theorem with respect to Σ ([2], [5], [6]) and that a distortion theorem of this type relating to four points cannot be obtained by using elementary functions as extremal functions. We use Jenkins' general coefficient theorem ([3], [4]) to prove our theorem and make a slight discussion to verify the extremal functions.

2. We now state the theorem.

THEOREM. *For all functions $g(z)$ belonging to Σ the inequalities*

$$\frac{(1-r^{-3})}{3\sqrt{3}\,r^3(1+r^{-3})^3}\leqq\frac{|g'(z)g'(z\omega)g'(z\omega^2)|}{|g(z)-g(z\omega)|\,|g(z\omega)-g(z\omega^2)|\,|g(z\omega^2)-g(z)|}\leqq\frac{(1+r^{-3})^3}{3\sqrt{3}\,r^3(1-r^{-3})^3}$$

hold where $z=re^{i\theta}$, $r>1$ and $\omega=e^{i2\pi/3}$. The left equality occurs only for the function $g(z)=z(1+e^{i3\theta}z^{-3})^{2/3}+k$ and the right only for the function $g(z)=z(1-e^{i3\theta}z^{-3})^{2/3}+k$ with k as an arbitrary constant.

Proof. We first prove the left inequality. We set $R_j=r(1+r^{-3})^{2/3}\omega^j$, $j=0,1,2$, and consider a quadratic differential

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