# ON THE RELATION BETWEEN THE DISTRIBUTIONS OF THE QUEUE SIZE AND THE WAITING TIME

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## § 1. Introduction.

In many articles on queuing theory, two measures of effectiveness, that is, queue size and waiting time are dealt with. However, it seems that their handlings are separated in many cases.

In this paper, we shall remark on the relation between the distribution of queue size and that of waiting time (especially the relation of the expectations). In some text books on operations research (e. g. [8]) by the rough and intuitive argument, the relation: $E(L)=\lambda E(W)$ is described, where $1/\lambda$ is the mean interarrival time, $E(W)$ is the expected waiting time and $E(L)$ is the mean queue size in the equilibrium state. And, the exact proof of this relation were done by calculating the both sides of this equality separately in some special cases. For instance, Morse [8] showed that the relation is valid in the cases $M/M/s$, $M/E_k/1$ and $E_2/M/1$.

We shall consider here four types of queue size in the equilibrium state which are denoted by $L$, $L^*$, $Q$ and $Q^*$ as follows:

$L$: queue size (not include the customer being served) observed at any time,

$L^*$: queue size observed at the epoch just before a customer arrives,

$Q$: queue size observed at the epoch just before the service of a customer begins,

$Q^*$: queue size observed at the epoch just after the service of a customer has finished.

In the above if we try to describe more exactly, (for example, to say about $L$), we must define $L$ as the random variable obeying to the limit distribution of $L(t)$ as $t\to\infty$, where $L(t)$ means the queue size at time $t$.

Throughout this paper we shall assume that $sE(X_i)>E(Y_i)$, which guarantees the existence of the limit distribution of $L(t)$, where $s$, $E(X_i)$ and $E(Y_i)$ mean the number of servers, the expected interarrival time and the expected service time, respectively.

Furthermore, this assumption will guarantee the existence of the equilibrium distribution (i. e., the limit distribution) of the other quantities mentioned in the above. (This facts were shown in [1] and [5].)

In the present paper we shall show that

( i ) the distribution of $L^*$ will be expressed using the distribution of the waiting time $W$ in the equilibrium state;

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