REMARKS CONCERNING TWO QUASI-FROBENIUS RINGS WITH ISOMORPHIC RADICALS

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The purpose of this short note is to make some supplementary remarks to the author's previous work [2] and refine theorem 2 of [2]. Let A and $\tilde{A}$ be two quasi-Frobenius rings and let the radical $\tilde{N}$ of $\tilde{A}$ be isomorphic to the radical N of A; we shall identify $\tilde{N}$ with N and say that A and $\tilde{A}$ have the same radical N. Let

$$A=\sum_{\kappa=1}^{k}\sum_{\iota=1}^{f(\kappa)}Ae_{\kappa,\iota}$$

be a decomposition of A into direct sum of indecomposable left ideals; the elemeats $e_{\kappa,\iota}$ $(1\leqq\kappa\leqq k,\ 1\leqq i\leqq f(\kappa))$ are mutually orthogonal primitive idempotents of A such that $Ae_{\kappa,\iota}\cong Ae_{\lambda,\jmath}$ if and only if $\kappa=\lambda$. We put $e_{\kappa,1}=e_{\kappa}$, $\sum_{\iota}e_{\kappa,\iota}=E_{\kappa}$; $E=\sum_{\kappa}E_{\kappa}$ is the unit element of A. Further, let $\tilde{e}_{\kappa,\iota}$, $\tilde{E}_{\kappa}$, etc. have the same meaning to $\tilde{A}$ as $e_{\kappa,\iota}$, E_{κ}, etc. to A. For a subset S of A we denote the left [right] annihilators of S by $l_A(S)$ $[r_A(S)]$; the notations $l_{\tilde{A}}(*)$, $l_N(*)$ etc. may be defined similarly.

Remembering theorem 1 of [2], we shall assume in this note that both A and $\tilde{A}$ are bound rings and that $M=l_N(N)=r_N(N)$ is contained in N^2. Then by theorem 2 of [2] $\bar{A}=A/N$ is isomorphic to $\bar{\tilde{A}}=\tilde{A}/N$; moreover, there is a (unique) 1-1 correspondence between the simple constituents of $\bar{A}$ and those of $\bar{\tilde{A}}$. So that we may assume, after a suitable reordering, that $\bar{A}_{\kappa}=\bar{A}\bar{E}_{\kappa}$ corresponds to $\bar{\tilde{A}}_{\kappa}=\bar{\tilde{A}}\bar{\tilde{E}}_{\kappa}$ in this correspondence $(1\leqq\kappa\leqq k)$.

Proposition 1. *Let A and $\tilde{A}$ be as above. Let $\mathfrak{l}\supset\mathfrak{l}'$ be two left A-ideals contained in N and let the factor module $\mathfrak{l}/\mathfrak{l}'$ be simple and isomorphic to Ae_{κ}/Ne_{κ}. Assume moreover that $\mathfrak{l}$ and $\mathfrak{l}'$ are left $\tilde{A}$-ideals. Then $\mathfrak{l}/\mathfrak{l}'$ is also a simple $\tilde{A}$-module and is isomorphic to $\tilde{A}\tilde{e}_{\kappa}/N\tilde{e}_{\kappa}$. Similarly for right ideals.*

Proof. First we assume that $\mathfrak{l}\smile M=\mathfrak{l}'\smile M$. Then we must have $\mathfrak{l}\frown M\supsetneqq\mathfrak{l}'\frown M$, and there exists a minimal left A-ideal $\mathfrak{l}_0$ in M such that $\mathfrak{l}\frown M=\mathfrak{l}'\frown M+\mathfrak{l}_0$; from this it follows that $\mathfrak{l}=\mathfrak{l}'+\mathfrak{l}_0$ and the assumption $\mathfrak{l}/\mathfrak{l}'\cong Ae_{\kappa}/Ne_{\kappa}$ shows that $\mathfrak{l}_0\cong Ae_{\kappa}/Ne_{\kappa}$. As $\mathfrak{l}_0$ is also a left $\tilde{A}$-ideal, we have that $\mathfrak{l}/\mathfrak{l}'\cong\mathfrak{l}_0\cong\tilde{A}\tilde{e}_{\kappa}/N\tilde{e}_{\kappa}$ is a simple $\tilde{A}$-module. Now assume that $\mathfrak{l}\smile M\supsetneqq\mathfrak{l}'\smile M$. In this case, $\mathfrak{l}\frown M$ must coincide with $\mathfrak{l}'\frown M$. In fact, if $\mathfrak{l}\frown M\supsetneqq\mathfrak{l}'\frown M$, we have for a suitable left A-ideal $\mathfrak{l}^*$ in M $\mathfrak{l}\frown M=\mathfrak{l}'\frown M+\mathfrak{l}^*$, which implies $\mathfrak{l}=\mathfrak{l}'+\mathfrak{l}^*$ since $\mathfrak{l}/\mathfrak{l}'$ is a simple A-module. This contradicts the assumption $\mathfrak{l}\smile M\supsetneqq\mathfrak{l}'\smile M$. Now, note that $\mathfrak{l}\smile M/\mathfrak{l}'\smile M=\mathfrak{l}\smile(\mathfrak{l}'\smile M)/\mathfrak{l}'\smile M\cong\mathfrak{l}/\mathfrak{l}'$ (as A-modules

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