POSITIVE HARMONIC FUNCTIONS ON AN END

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It is well known that the Martin theory on positive harmonic functions plays an important role in the theory of open Riemann surfaces. Its whole theory depends upon the potential theory and the so-called Martin compactification of the given surface. In the present paper we shall give a proof of it, especially the representation theorem on positive harmonic functions on an end, which seems extremely simple. In our proof we shall introduce a suggestive functional and make use of a variational method.

Let W be an open Riemann surface and $\{W_m\}$ be its exhaustion in the usual sense. Let $HP(W-\overline{W}_m)$ be a class of positive harmonic functions on $W-\overline{W}_m$ vanishing continuously on ∂W_m. Evidently $HP(W-\overline{W}_m)$ is a positively linear space.

LEMMA 1. *The space $HP(W-\overline{W}_m)$ is a metric space with the metric*

$$\rho(u, v)=\int_{\partial W_m} \frac{\partial}{\partial n}|u-v|\,ds.$$

LEMMA 2. *The uniform convergence in the wider sense in $W-\overline{W}_m$ in the class $HP(W-\overline{W}_m)$ is equivalent to the ρ-convergence in the space $HP(W-\overline{W}_m)$.*

LEMMA 3. *The unit sphere U_P in the space $HP(W-\overline{W}_m)$ is a ρ-compact convex set.*

Proof. Let $v\in U_P$, then

$$\int_{\partial W_m} \frac{\partial}{\partial n} v(p)\,ds=1.$$

Let $\omega_q(p)$ be the harmonic measure $\omega(p, \partial W_q, W_q-\overline{W}_m)$ and $M=\max v(p)$, $d=\min v(p)$ on ∂W_q. Then we have $d\omega_q(p)\leqq v(p)$ on $W_q-\overline{W}_m$ and hence on ∂W_m

$$d\frac{\partial}{\partial n}\omega_q(p)\leqq \frac{\partial}{\partial n}v(p).$$

Let l denote the value of the integral

$$\int_{\partial W_m} \frac{\partial}{\partial n}\omega_q(p)\,ds,$$

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