A REMARK ON THE EXPECTATIONS OF OPERATOR ALGEBRAS

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If a transformation $a \to a^{\varepsilon}$ defined on a C^*-algebra having the identity, which mapps it into itself, satisfies; i) linear, ii) indempotent: $a^{\varepsilon\varepsilon} = a^{\varepsilon}$, iii) positive: $a^{\varepsilon} \geqq 0$ for $a \geqq 0$, iv) unit-preserving: $1^{\varepsilon} = 1$, and moreover $(x^{\varepsilon}y)^{\varepsilon} = x^{\varepsilon}y^{\varepsilon}$, then ε is said to be an *expectation* in the sense of [7]. Clearly, the notion of the expectation is an extension of that of traces, states, Dixmier's centering $\natural$ etc. The endomorphism of Reynolds [5], which is defined on a functional algebra, is also an expectation. A detailed treatise on expectations of discrete finite factors will be found in a recent publication of Davis [3].

Among expectation, in a connection with the theory of probability, the conditional expectation in the sense of [11] has the special importance. In a von Neumann algebra A with a faithful normal trace τ, for a von Neumann subalgebra B of A, the *conditional expectation* ε *conditioned by* B is defined as an operation which maps $a \in A$ to $a^{\varepsilon} \in B$ such that $\tau(ab) = \tau(a^{\varepsilon}b)$ for any $b \in B$. It is known that the conditional expectation in the sense of the above coincides with the usual one if the algebra A is commutative (e.g., cf. [7] or [11]). It is also obvious that the conditional expectation is a projection which maps A onto B considering A as a pre-Hilbert space introducing an inner product by the trace as usually.

The present note contains a proof that the conditional expectation in a not necessarily commutative probability is completely positive in the sense of Stinespring [8] or positive definite in the sense of [10]. Consequently two representation theorems on the conditional expectation follow. They will be proved in §1, and their generalizations on C^*-algebras will be discussed in §3 briefly.

The remainder of the note, §2, contains a discussion to generalize Jensen's inequality for non-commutative probability. A formal extension of the operator convexity of Bendat-Sharman [1] allows us a formulation for finite continuous factors, by the help of Davis' theorem [2]. However, the final form still remains for.

1. *Complete positivity of the conditional expectation.*

A linear transformation θ defined on a C^*-algebra A into an operator algebra

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