ON COEFFICIENT-REGIONS OF LAURENT SERIES WITH POSITIVE REAL PART

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1. Introduction.

Let $\mathfrak{R}_0 = \{\Phi(z)\}$ be the class of analytic functions which are regular and of positive real part in the unit circle $|z| < 1$ and normalized by $\Phi(0) = 1$. It is well known that Carathéodory [1, 2] has established a result on the variability-region of Taylor coefficients for $\Phi(z) \in \mathfrak{R}_0$; cf. also Rogosinski [6].

We now consider, besides $\mathfrak{R}_0$, the class $\mathfrak{R}_q = \{\Phi(z)\}$ consisting of analytic functions which are single-valued, regular and of positive real part in an annulus $(0 <) q < |z| < 1$ and normalized by the conditions

$$\Re\Phi(z) = 1 \text{ along } |z| = q \quad \text{and} \quad \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi(qe^{i\theta})\, d\theta = 1.$$

Let the Laurent expansion of a function $\Phi(z) \in \mathfrak{R}_q$ be

$$\Phi(z) = 1 + \sum_{\nu=-\infty}^{\infty}{}' c_\nu z^\nu \qquad (q < |z| < 1)$$

where the prime means that the summation extends over all integers except $\nu = 0$. Then, for any two positive integers m and n, the point $\mathrm{P} = \mathrm{P}_{-m}^{n}[\Phi]$ with the coordinates $\{c_\nu;\ -m \leqq \nu \leqq n,\ \nu \neq 0\}$ in the complex $(n+m)$-dimensional space is called the *$(-m, n)$th coefficient-point* of $\Phi(z)$. The purpose of the present paper is to determine precisely the *variability-region*, that is, the range of the point-set consisting of all possible points $\mathrm{P}_{-m}^{n}[\Phi]$ when $\Phi(z)$ extends over the class $\mathfrak{R}_q$. The class $\mathfrak{R}_0$ is, of course, regarded as a limiting case where the interior boundary component of the annulus degenerates to a single point, i.e. the origin.

On the other hand, if the first normalization for $\mathfrak{R}_q$ that $|z| = q$ corresponds to a segment parallel to the imaginary axis is rejected, there occurs an extended class $\hat{\mathfrak{R}}_q$ which includes $\mathfrak{R}_q$ as a subclass. Namely, let $\hat{\mathfrak{R}}_q = \{\Phi(z)\}$ denote the class of analytic functions which are single-valued, regular and of positive real part in the annulus $(0<) q < |z| < 1$ and normalized by the condition

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi(re^{i\theta})\, d\theta = 1 \qquad (q < r < 1).$$

Corresponding to the case of $\mathfrak{R}_q$, we shall consider also an analogous problem with respect to this extended class.

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