AFFINE CONNEXIONS IN AN ALMOST PRODUCT SPACE

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Let us consider an almost complex space M of class C^∞ and denote by $F_\lambda{}^\kappa$ its structure tensor satisfying $F_\mu{}^\lambda F_\lambda{}^\kappa = -A_\mu^\kappa$. As is well known now, in order that the tensor $F_\lambda{}^\kappa$ define a complex structure, it is necessary and sufficient that the Nijenhuis tensor

$$N_{\mu\lambda}{}^\kappa = F_\mu{}^\rho(\partial_\rho F_\lambda{}^\kappa - \partial_\lambda F_\rho{}^\kappa) - F_\lambda{}^\rho(\partial_\rho F_\mu{}^\kappa - \partial_\mu F_\rho{}^\kappa)$$

constructed from $F_\lambda{}^\kappa$ vanish. (Eckmann [1], Eckmann and Frölicher [1], Ehresmann [1], Frölicher [1], Libermann [1], [2], [3], Newlander and Nirenberg [1], Nijenhuis [1], de Rham (unpublished), Yano [2], [3].)

If there exists a symmetric affine connexion $\Gamma_{\mu\lambda}^\kappa$ in M, then denoting by ∇_μ the covariant derivative with respect to this affine connexion, we have

$$N_{\mu\lambda}{}^\kappa = F_\mu{}^\rho(\nabla_\rho F_\lambda{}^\kappa - \nabla_\lambda F_\rho{}^\kappa) - F_\lambda{}^\rho(\nabla_\rho F_\mu{}^\kappa - \nabla_\mu F_\rho{}^\kappa)$$

and consequently we can see that if there exists a symmetric affine connexion such that $\nabla_\mu F_\lambda{}^\kappa = 0$, then $N_{\mu\lambda}{}^\kappa = 0$ and consequently the almost complex structure is a complex structure.

Now the eigenvalues of the matrix $(F_\lambda{}^\kappa)$ are $+i$ and $-i$ and the eigenvectors corresponding to the eigenvalue $+i$ span a distribution B of complex dimension n and those corresponding to the eigenvalue $-i$ span a distribution $\bar{B}$ which is complex conjugate to B. The condition $\nabla_\mu F_\lambda{}^\kappa = 0$ means then that these two complex conjugate distributions are parallel with respect to the symmetric affine connexion. (Yano [2].)

Now the following converse problem arises. We assume that $N_{\mu\lambda}{}^\kappa = 0$. Then does there exist a symmetric affine connexion $\Gamma_{\mu\lambda}^\kappa$ such that the covariant derivative $\nabla_\mu F_\lambda{}^\kappa$ of the structure tensor $F_\lambda{}^\kappa$ vanishes? This problem was studied by Eckmann [1] and Frölicher [1] and answered affirmatively. (Cf. Yano [3].)

Now problems quite analogous to this arise in a space which we call here an almost product space. Suppose that there are given two complementary distributions B and C of respective dimensions p and q ($p \geqq 1$, $q \geqq 1$, $p+q=n$), then denoting by B_λ^κ and C_λ^κ the projection tensors on these distributions, we have

$$B_\lambda^\kappa + C_\lambda^\kappa = A_\lambda^\kappa.$$

It is easy to verify that if we put

$$B_\lambda^\kappa - C_\lambda^\kappa = F_\lambda{}^\kappa,$$

Received October 11, 1958.