ON CONFORMAL MAPPING OF A DOMAIN WITH CONVEX OR STAR-LIKE BOUNDARY

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1. Introduction.

A necessary and sufficient condition for an analytic function $f(z)$ to be regular in the unit circle $E: |z| < 1$ and to map E univalently onto a convex domain is that it satisfies

$$\Re\left(1 + z\frac{f''(z)}{f'(z)}\right) > 0 \qquad \text{for} \quad |z| < 1.$$

This is a well-known classical theorem originally due to Study [11]. Its sufficiency proof particularly with respect to the regularity and univalency of $f(z)$ has been later supplemented by Kobori [4]. Once these properties of $f(z)$ having been established, it is ready to show that the convexity of the image-domain $f(E)$ follows from the condition of the theorem. In fact, in view of the relation

$$\frac{d}{d\varphi}\arg df(re^{i\varphi}) = \Re\left(1 + re^{i\varphi}\frac{f''(re^{i\varphi})}{f'(re^{i\varphi})}\right) > 0$$

for any fixed r with $0 \leqq r < 1$, the image-domain of any concentric circular disc $|z| < r(< 1)$ by $f(z)$ is convex so that $f(E)$ is itself convex.

On the other hand, Carathéodory [3] has given a proof in which the necessity of the condition is shown by making use of a convergence theorem on variable domains established by himself [2]. Later Radó [10] has given a very elementary proof of the fact that if $f(z)$ maps the whole circle $|z| < 1$ univalently onto a convex domain then it maps every concentric circle also onto a convex domain. The necessity part of Study's theorem may be regarded as its immediate consequence. In fact, there then holds

$$\Re\left(1 + z\frac{f''(z)}{f'(z)}\right) = \frac{d}{d\varphi}\arg df(re^{i\varphi}) \geqq 0 \qquad (z = re^{i\varphi})$$

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