FOURIER SERIES XI: GIBBS' PHENOMENON

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1. Concerning the Gibbs phenomenon of Fourier series, H. Cramér [1] proved the following

THEOREM 1. *There exists a number r_0, $0 < r_0 < 1$, with the following property: If $f(x)$ is simply discontinuous at a point ξ, the (C, r) means $\sigma_n^r(x)$ of the Fourier series of $f(x)$ present Gibbs' phenomenon at ξ for $r < r_0$, but not for $r \geqq r_0$.*

We shall extend this theorem to the discontinuity of the second kind. In this direction S. Izumi and M. Satô [2] proved the the following

THEOREM 2. *Suppose that*

$$f(x) = a\psi(x-\xi) + g(x), \tag{1}$$

where $\psi(x)$ is a periodic function with period 2π such that

$$\psi(x) = (\pi - x)/2 \qquad (0 < x < 2\pi) \tag{2}$$

and where

$$\begin{aligned} &\limsup_{x\downarrow\xi}\ g(x) = 0, \qquad &&\liminf_{x\uparrow\xi}\ g(x) = 0, \\ &\liminf_{x\downarrow\xi}\ g(x) \geqq -a\pi, \qquad &&\limsup_{x\uparrow\xi}\ g(x) \leqq a\pi, \end{aligned} \tag{3}$$

$$\int_0^x |g(\xi+u)|\,du = o(|x|), \tag{4}$$

then the Gibbs' phenomenon of the Fourier series of $f(x)$ appears at $x = \xi$.

We shall prove that Theorem 1 holds even when ξ is the discontinuity point of the second kind, satisfying the condition in Theorem 2. More precisely,

THEOREM 3. *Suppose that*

$$f(x) = a\psi(x-\xi) + g(x), \tag{1}$$

where $\psi(x)$ is a periodic function with period 2π such that

$$\psi(x) = (\pi - x)/2 \qquad (0 < x < 2\pi) \tag{2}$$

and where

$$\begin{aligned} &\limsup_{x\downarrow\xi}\ g(x) = 0, \qquad &&\liminf_{x\uparrow\xi}\ g(x) = 0, \\ &\liminf_{x\downarrow\xi}\ g(x) \geqq -a\pi, \qquad &&\limsup_{x\uparrow\xi}\ g(x) \leqq a\pi, \end{aligned} \tag{3}$$

$$\int_0^x |g(\xi+u)|\,du = o(|x|). \tag{4}$$

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