ON A NON-NEGATIVE SUBHARMONIC FUNCTION IN A HALF-PLANE

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1. We shall prove

THEOREM 1. *Let* $u(z) = u(x+iy) \not\equiv 0$ *be a non-negative subharmonic function in a half-plane* $x > 0$, *which vanishes continuously on the imaginary axis.*
Let

$$m(r) = m(r, u) = \int_{-\pi/2}^{\pi/2} u(re^{i\theta}) \cos\theta \, d\theta, \qquad 0 < r < \infty,$$

then
(i) $m(r)/r$ *is a continuous non-decreasing function of* r *and is a convex function of* $1/r^2$. *Hence*

$$\lim_{r\to\infty} \frac{m(r)}{r} = c, \qquad 0 < c \leqq \infty,$$

exists.
If $0 < c < \infty$, *then*

(ii) $$u(z) = kx - \int_{\Re(a)>0} \log\left|\frac{z+\bar{a}}{z-a}\right| d\mu(a), \qquad k = \frac{2c}{\pi},$$

where μ *is a positive mass distribution in* $x > 0$, *such that*

$$\int_{\Re(a)>0} \frac{\Re(a)}{|a|^2} d\mu(a) < \infty.$$

(iii) *Except a set of* θ *of logarithmic capacity zero,*

$$\lim_{r\to\infty} \frac{u(re^{i\theta})}{r} = k\cos\theta$$

exists.

That $m(r)/r$ is a non-decreasing function of r is proved by Ahlfors [1] and the proof is simplified by Dinghas [2]. (iii) is proved by Ahlfors and Heins [3].
As a special case, we have

THEOREM 2. *Let* $f(z)$ *be regular in* $x > 0$ *and continuous and* $|f(z)| \leqq 1$ *on the imaginary axis. Suppose that* $\log^+|f(z)| \not\equiv 0$ *and let*

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