ABEL SUMMABILITY OF DERIVED CONJUGATE FOURIER SERIES

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The Abel summability of the derived conjugate series has been discussed by Plessner [4], Moursund [3] and Misra [2]. Moursund's result is very complicated and Misra proved a simpler theorem, but it is not general. The object of this note is to prove a simpler and more general theorem. In §1, we shall prove a summability theorem of the conjugate series. This is another result of Misra [1], and our method of the proof is simpler than Misra's. In §2, we shall reduce the summability theorem of the derived conjugate series to the case of §1.

1. Let f(x) be an integrable and periodic function and

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx),$$

$$\psi(x,t) \equiv \psi^{(0)}(x,t) = \frac{1}{2}\{f(x+t) - f(x-t)\}$$

$$\sim \sum_{n=1}^{\infty}(b_n \cos nx - a_n \sin nx)\sin nt$$

$$\equiv \sum_{n=1}^{\infty} B_n(x) \sin nt.$$

Since $t/(1+t^2)$ is of bounded variation in $(0,\infty)$ and tends to zero as $t\to\infty$, we have for any fixed $\varepsilon > 0$

$$\int_0^{\infty} \psi(x,t)\,\frac{t/\varepsilon}{1+(t/\varepsilon)^2}\,dt$$

$$= \sum_{n=1}^{\infty} B_n(x) \int_0^{\infty} \frac{t/\varepsilon}{1+(t/\varepsilon)^2} \sin nt\,dt$$

$$= \frac{\pi\varepsilon}{2}\sum_{n=1}^{\infty} B_n(x)\, e^{-\varepsilon n}.$$

The Abel mean of $\sum B_n(x)$ is

$$V(x,\varepsilon) \equiv \sum_{n=1}^{\infty} B_n(x) e^{-\varepsilon n}$$

$$= \frac{2}{\pi}\int_0^{\infty}\frac{t}{\varepsilon^2+t^2}\psi(x,t)\,dt$$

$$= \frac{2}{\pi}\int_0^{\infty}\psi(x,t)\,\bar{P}(\varepsilon,t)\,dt,$$

say. We denote by

$\psi_n(x,t)$ the n-th integral of $\psi(x,t)$, then

$$|\psi_n(x,t)| \leqq M t^{n-1}, \quad \text{as} \quad t\to\infty$$

Since for $n = 0, 1, 2, \dots$

$$\text{(a)} \quad \frac{\partial^n \bar{P}(\varepsilon,t)}{\partial t^n} = O(\varepsilon^{-(n+1)}) \quad (t \leqq \varepsilon),$$

$$\text{(b)} \quad \frac{\partial^n \bar{P}(\varepsilon,t)}{\partial t^n} = O(t^{-(n+1)}) \quad (t\to\infty),$$

we have, by successive partial integration,

(1) $V(x,\varepsilon)$

$$= \frac{2}{\pi}\Big[\psi_1(x,t)\bar{P}(\varepsilon,t)\Big]_0^{\infty} - \frac{2}{\pi}\int_0^{\infty}\psi_1(x,t)\frac{\partial \bar{P}(\varepsilon,t)}{\partial t}dt$$

$$= -\frac{2}{\pi}\int_0^{\infty}\psi_1(x,t)\frac{\partial \bar{P}(\varepsilon,t)}{\partial t}dt$$

$$= (-1)^n\frac{2}{\pi}\int_0^{\infty}\psi_n(x,t)\frac{\partial^n \bar{P}(\varepsilon,t)}{\partial t^n}dt.$$

Let us put

$$\bar{P}(\varepsilon,t) = \frac{1}{t} - \frac{\varepsilon^2}{t(\varepsilon^2+t^2)} = \frac{1}{t} - Q(\varepsilon,t),$$

then

$$\text{(c)} \quad \frac{\partial^n Q(\varepsilon,t)}{\partial t^n} = O(\varepsilon^2 t^{-(n+3)}) \quad \text{as} \quad t\to\infty.$$