ON TRANSFERENCE BETWEEN BOUNDARY VALUE PROBLEMS FOR A SPHERE

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1. Introduction.

In two-dimensional case it has been shown that two kinds of boundary value problems, Dirichlet and Neumann problems, are readily transferable each other for some domains of simple elementary configuration. Such a method for transference has been availed by Myrberg for a circular disc,[1] and subsequently discussed by one of the present authors for certain simply-connected slit domains[2] and for an annulus[3]

In the present Note we shall show that the method remains valid also for a sphere of any higher dimension. Since the Poisson integral formula for solving Dirichlet problem is classical, our result will offer an elementary construction of an integral formula for solving Neumann problem for a sphere.

In an N-dimensional euclidean space, the rectangular cartesian coordinates $(x_1, \ldots, x_N)$ and the polar coordinates $(r, \vartheta_1, \ldots, \vartheta_{N-1})$ are connected by the relations[4]

$$x_j = r\cos\vartheta_j \prod_{k=1}^{j-1} \sin\vartheta_k \qquad (1 \leq j \leq N-1),$$

$$x_N = r \prod_{k=1}^{N-1} \sin\vartheta_k ;$$

$$r \geqq 0,$$

$$0 \leqq \vartheta_j \leqq \pi \qquad (1 \leqq j \leqq N-2),$$

$$0 \leqq \vartheta_{N-1} < 2\pi ;$$

here and henceforth the empty product is always understood, as usual, to denote unity. The square of the line element is given by

$$ds^2 = \sum_{j=1}^{N} dx_j^2 = dr^2 + r^2 \sum_{j=1}^{N-1} d\vartheta_j^2 \prod_{k=1}^{j-1} \sin^2\vartheta_k$$

and the volume element is expressed in the form

$$d\tau = \prod_{j=1}^{N} dx_j = r^{N-1} dr \prod_{j=1}^{N-1} \sin^{N-j-1}\vartheta_j \, d\vartheta_j .$$

The Laplacian operator is represented by

$$\Delta = \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} = \frac{1}{r^{N-1}} \frac{\partial}{\partial r}\left(r^{N-1}\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\Delta^*$$

where Δ^* denotes a self-adjoint differential operator defined by

$$\Delta^* = \sum_{j=1}^{N-1}\left(\prod_{k=1}^{j-1}\operatorname{cosec}^2\vartheta_k\right) \cdot \operatorname{cosec}^{N-j-1}\vartheta_j \frac{\partial}{\partial\vartheta_j}\left(\sin^{N-j-1}\vartheta_j \frac{\partial}{\partial\vartheta_j}\right).$$

The self-adjoint partial differential equation

$$\Delta^* Y + \lambda Y = 0$$

defines a spherical surface harmonic Y_n of order n as a solution belonging to the eigen-value $n(n+N-2)$.[5] Its general form is a linear combination of $\binom{n+N-1}{N-1} - \binom{n+N-3}{N-1}$ functions which constitute a linearly independent basis.

2. Lemmas.

We begin with a lemma involving a formal identity.