ON AN INEQUALITY CONCERNING THE EIGENVALUE PROBLEM OF MEMBRANE

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Let D be a plane domain with a given area A surrounded by a boundary C. We consider the eigenvalue problem on the equation for a vibrating membrane

$$\Delta u + \lambda u = 0, \qquad \lambda > 0,$$

under the boundary condition

$$u = 0 \quad \text{on } C.$$

The principal frequency of a membrane is diminished by symmetrization with respect to a straight line as well as to a point, and hence, for a given area the principal frequency attains its minimum for a circle. On the other hand, the second frequency does not behave similarly. Pólya and Szegö have pointed out that a rectangle with sides a and $2a$ has the second frequency less than that of a circle with the same area.

In the present paper we shall study the greatest lower bound for the second frequency of all membranes having a given area. Our result may be stated as follows:

Proposition. The second frequency of all membranes, having a given area A and fixed along their boundaries, has the greatest lower bound equal to the principal frequency of a circle having the area $A/2$.

Before giving a proof of our proposition, we start with observing some fundamental properties of the second eigenfunction. Let u_1, u_2 be the first and the second eigenfunctions respectively, and λ_1, λ_2 be the corresponding eigenvalues. It is well known that the second eigenfunction u has necessarily a nodal line in the interior of the domain D. Precisely, the domain should be divided into two parts D' and D'' by a nodal line σ, each of them is a connected domain. u is positive in one of D' and D'' and negative in the other. The boundary C is divided into two parts C' and C'' in such a manner. C' and the nodal line σ surround the domain D' while C'' and σ the domain D''. Then u_2 is a function which satisfies the equation

$$\Delta u + \lambda_2 u = 0 \qquad \text{in } D'$$

and vanishes on $C' + \sigma$, and simultaneously satisfies

$$\Delta u + \lambda_2 u = 0 \qquad \text{in } D''$$

and vanishes on $C'' + \sigma$. Moreover u_2 vanishes neither in the interior of D' nor in that of D''. It is to be noticed that, if a solution of the equation $\Delta u + \lambda u = 0$ under the vanishing boundary condition, never vanishes in the interior of the domain, then it must necessarily be the first eigenfunction. Therefore our second eigenfunction u_2 may be regarded as the first eigenfunction of the domain D' as well as of the domain D'', and accordingly λ_2 can be regarded as the first eigenvalue both of D' and D''.

We are now in the position to give a proof for our proposition. Let μ' be the first eigenvalue of the circle with the same area as D', and μ'' be that of the circle with the same area as D''.

According to the fundamental inequality for the first eigenvalue, we can obtain two inequalities

$$\lambda_2 \geq \mu', \qquad \lambda_2 \geq \mu'',$$

which hold simultaneously, hence follows

$$\lambda_2 \geq \mathrm{Max}(\mu', \mu'').$$