ON DECOMPOSITIONS OF A COMMUTATIVE SEMIGROUP

By Takayuki TAMURA and Naoki KIMURA

If there exists a homomorphism of a semigroup S onto a semigroup S^* having special type, all elements of S are decomposed into the class sum of mutually disjoint subsets. Then we say that the decomposition of S to S^* is obtained. In particular the decomposition to a semilattice is of importance, i.e., $S = \bigcup_{\alpha\in\Gamma} S_\alpha$ where $S_\alpha \cap S_\beta = \phi$ $(\alpha \neq \beta)$, every S_α is a restrictive subsemigroup, and for any α, β, there is a unique γ such that $S_\alpha S_\beta \subset S_\gamma$ as well as $S_\beta S_\alpha \subset S_\gamma$. In §1 we argue that there is greatest decomposition of a semigroup to a semilattice; particularly in §2 we show a decomposition of a commutative semigroup by method different from Mr. Numakura's, and in §3 our decomposition is proved to be greatest.

§1 Greatest decomposition

In this paragraph S is assumed to be a general semigroup. A decomposition of S to an idempotent semigroup gives an equivalence relation; and an equivalence relation $\sim$ in S raises a decomposition of S to an idempotent semigroup if and only if

(1) $a \sim b$, $c \sim d$ imply $ac \sim bd$,
(2) if $a \sim b$ then $a \sim ab$.

Lemma 1. (1) and (2) are equivalent to (1') and (2'),

(1') $a \sim b$ implies $ac \sim bc$ and $ca \sim cb$ for every c,
(2') $a \sim a^2$ for every a.

Proof. (1') $\rightarrow$ (1): For, from $a \sim b$, follows $ac \sim bc$; and from $c \sim d$, $bc \sim bd$. By transitivity, $ac \sim bd$. (1) $\rightarrow$ (1'): evident. (1') & (2') $\rightarrow$ (2): from $a \sim b$, it follows that $a \sim a^2 \sim ab$. (2) $\rightarrow$ (2'): evident.

We denote by $\mathfrak{D}$ the set of all decompositions φ of S to a semilattice, and by $\overset{\varphi}{\sim}$ the congruence relation which gives φ. The relation $\overset{\varphi}{\sim}$ and $\overset{\psi}{\sim}$ are equal if they give the same decomposition. Obviously $\mathfrak{D}$ is not empty, because it includes at least a trivial decomposition, a partition of all elements of S into one class.

Now we introduce the ordering into $\mathfrak{D}$: i.e. $\varphi \geqq \psi$ means that $x \overset{\psi}{\sim} y$ if $x \overset{\varphi}{\sim} y$. The ordering is clearly a partial ordering. Then we have the below lammas.

Lemma 2. $\mathfrak{D}$ forms a complete semilattice.

Proof. Since $\mathfrak{D}$ is a partly ordered set, we show that any subset $\mathfrak{D}_1$ of $\mathfrak{D}$ has a least upper bound. We define a relation $\overset{\xi}{\sim}$ as follows. $x \overset{\xi}{\sim} y$ means that $x \overset{\varphi}{\sim} y$ for every $\varphi \in \mathfrak{D}_1$. It is not hard to verify that $\overset{\xi}{\sim}$ is an equivalence relation and satisfies the condition (1') and (2') (in Lemma 1). Clearly $\xi \geqq \varphi$ for all $\varphi \in \mathfrak{D}_1$. Take up any $\xi' \geqq \varphi$ for all $\varphi \in \mathfrak{D}_1$, then from $x \overset{\xi'}{\sim} y$ follows $x \overset{\varphi}{\sim} y$ for all $\varphi \in \mathfrak{D}_1$, i.e., $x \overset{\xi}{\sim} y$; hence $\xi' \geqq \xi$, and so ξ is the least upper bound of $\mathfrak{D}_1$. Consequently

Theorem 1. There is a greatest element of $\mathfrak{D}$. In other words, there exists the greatest decomposition of a semigroup to a semilattice.

In another article we shall relate what is an equivalence relation giving the greatest decomposition of a general semigroup.

§2 A decomposition of a commutative semigroup

Let S be a commutative semigroup. We define an ordering $a \geqq b$ between elements a and b of S to mean that a certain element $x \in S$ and a positive integer m are found such that

$$a^m = bx$$