NOTE ON UNIPOTENT INVERSIBLE SEMIGROUPS[1]

By Takayuki TAMURA

A semigroup with only one idempotent is called unipotent [2]. In this note we shall investigate the construction of unipotent inversible semigroup (defined as below). After all the study of such a semigroup will be reduced to that of a zero-semigroup [3].

Lemma 1. A semigroup is unipotent if and only if it contains the greatest group [4].

Proof. Suppose that a semigroup S has its greatest group G, and S contains idempotents e and f. Then, since $\{e\}$ and $\{f\}$ are groups in S, we see that $\{e\} \subset G$ and $\{f\} \subset G$; e and f are idempotents contained in G. Hence $e = f$; S is unipotent. Conversely, if S is unipotent, S has at least one group as a subsemigroup. Let $\{G_\alpha\}$ $(\alpha \in \Gamma)$ be the set of all groups in S. Since every G_α has the idempotent e of S in common, the semigroup G generated by all G_α $(\alpha \in \Gamma)$ is proved to be a group. It is easy to see that G is greatest.

When a unipotent semigroup S, for example, is finite, the greatest group G is represented as $G = Se$ where e is an idempotent. What is the necessary and sufficient condition in order that Se is the greatest group of S ?

Let S be a unipotent semigroup with an idempotent e. If, for any $a \in S$, there exists $b \in S$ such that $ab = e$ $(ba = e)$, S is called right (left) inversible, and b is a right (left) inverse of a. Of course b depends on a. Then since e is a right (left) zeroid [5] of S, a unipotent right (left) inversible semigroup is equivalent to a unipotent semigroup with zeroids [5]. The following lemmas follow immediately from the general theories of a semigroup with zeroids.

Lemma 2. Let S be a unipotent semigroup. The following conditions are equivalent.

(1) S is right inversible.
(2) S is left inversible.
(3) Se is a group.
(4) eS is a group.

We need no distinction between right inversibleness and left inversibleness. If S is right or left inversible, it is said to be inversible.

Lemma 3. Let S be a unipotent inversible semigroup, and G be its greatest group.

(1) $G = Se = eS$
(2) G is a two-sided ideal of S as well as the least one-sided ideal of S.
(3) e commutes with every $x \in S$.
(4) S is homomorphic on G by the mapping $\varphi(x) = xe = ex$.

We denote by Z the difference semigroup of S modulo G [6]. Z is a zero-semigroup.

Now we shall discuss the structure of a semigroup with zeroids in preparation for the theory of a unipotent inversible semigroup.

Let S be a semigroup having zeroids, and U be its group of zeroids. Since U is a two-sided ideal, we can consider the difference semigroup M of S modulo U; and M is a semigroup with a zero. Conversely, if we are given arbitrarily a semigroup M with a zero and a group U disjoint from M, there exists always at least one ramified homomorphism[7] ψ of M into U, e.g., the mapping of all non-zero elements of M into the unit of U. Consequently we have the following lemma[7].

Lemma 4. Given a semigroup M with a zero 0, and a non-trivial group U which is disjoint from M, and given a ramified homomorphism ψ of M into U, we can construct uniquely a semi-