MAXIMAL SUBGROUPS OF A SEMIGROUP

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The purpose of the present paper is to clarify the structure of two families made up of all unitary subsemigroups and of all subgroups of a semigroup. Above all the most important result is the existence theorem on maximal subgroups of a semigroup.

By a semigroup is meant a set M of elements a, b, c, . . . closed under an associative multiplication, i.e.,

$a, b \in M$ implies $ab \in M$, and

$(ab)c = a(bc)$.

A subset S of a semigroup M is called a subsemigroup, if S is closed under the multiplication. An element u of M is called an idempotent element if $u^2 = u$, and is called a unit if $ux = xu = x$ for any element x of M. A semigroup with a unit is called unitary. In what follows M is always to be understood as a semigroup with at least one idempotent element, and the totality of all the idempotent elements of M is denoted by I.

§1. Maximal unitary subsemigroups.

A unit of a semigroup is obviously an idempotent element, and a semigroup can not possess more than one unit.

Lemma 1. If $u \in I$, then $uMu = \{uau; a \in M\}$ is the greatest subsemigroup of M having u as a unit.

Proof. An element a of M belongs to uMu, if and only if $a = uau$. Thus u belongs to uMu, for $uuu = u^3 = u$.

For any two elements a, b of uMu, $ab = (uau)(ubu) = u(auub)u \in uMu$. From this follows that uMu is a subsemigroup.

u is clearly a unit of uMu, for

$u(uau) = (uau)u = uau$.

Next let S be a unitary subsemigroup of M with u as a unit. Then for any element a of S, we have $uau = a$. Therefore S is a subset of uMu. This shows that uMu is the greatest unitary subsemigroup of M with u as a unit.

We write $u \geqq v$ for two idempotent elements $u, v \in I$, if $uvu = v$. By this order I forms a partly ordered set.

Obviously we have $uMu \supset vMv$, if and only if $u \geqq v$. Therefore we have the following

Theorem 1. Let $\mathfrak{S}$ be the totality of the maximal unitary subsemigroups of M. Then $\mathfrak{S}$ forms a partly ordered set by set inclusion which is order-isomorphic with I.

§2. The greatest subgroup of a unitary semigroup.

In this § 2, let M be a unitary semigroup, and let u be the unit.

An element a of a semigroup M is called adversible, if $aM = Ma = M$. The set of all adversible elements of M is denoted by M*. It is to be noted that M* is not empty because u belongs to M*.

Lemma 2. M* is a subgroup of M.

Proof. M* includes the unit u. For any element $a \in M^*$ we can find two elements $x, y \in M$ such that

$$ax = ya = u,$$

in view of adversibility of a.

It follows that such elements x, y are the same and unique, since we have

$$x = ux = (ya)x = y(ax) = yu = y.$$

This element is denoted as a'.