ON A MAXIMALITY OF A CLASS OF POSITIVE HARMONIC FUNCTIONS

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A definition of harmonic dimension for any extended C-end has been given in our previous paper [1]. A principal aim of the present note is to establish that a class of positive harmonic functions with some restrictions is maximal in certain sense and to give another but equivalent definition of harmonic dimension for any extended C-end which is a natural consequence of the maximality. This new formulation is more convenient to the various purposes and more intrinsic in some senses than the former one.

1. Let Ω be an extended C-end having Γ as its non-compact analytic relative boundary. (Cf. Ozawa [1].) Let $g(z,p_m)$ be the Green function of Ω with pole at p_m. The harmonic dimension $\dim(\Omega)$ or $CH(\Omega)$ of Ω means a maximal cardinal number of linearly independent limit functions $\lim_{m\to\infty} g(z,p_m)$ which is non-trivial on Ω, where the limiting process $m\to\infty$ is taken along a suitable non-compact sequence $\{p_m\}$. Let G_Ω be a set of any linear combinations of such limit functions, with positive coefficients, each element of which is positive on Ω.

Let Q_Ω be a family of positive harmonic function w on Ω, vanishing identically on Γ, and subjecting to a condition

$$0 < \int_\Gamma \frac{\partial}{\partial\nu} w\, ds < \infty.$$

Let $\hat{\Omega}$ —— an end in Heins' sense —— mean a doubled domain of Ω, symmetric with regard to $\Gamma-\gamma$, γ being a compact part of Γ. $\widetilde{\square}$ means, in general, the symmetric configuration of a configuration $\square$ with respect to $\Gamma-\gamma$. Then $\hat{\Omega}=\Omega+\widetilde{\Omega}+(\Gamma-\gamma)$. Let $P_{\hat{\Omega}}$ be a family of positive harmonic functions on $\hat{\Omega}$ with vanishing boundary value on $\gamma+\widetilde{\gamma}$. Let $\{F_n\}_{n=0,1,\cdots}$ be an exhaustion of symmetric surface F into which $\hat{\Omega}$ is imbedded such that $F_0 = F-\bar{\hat{\Omega}}$ is compact and has $\gamma+\widetilde{\gamma}$ as its compact relative boundary. Here F and F_n are supposed to be symmetric with respect to $\Gamma-\gamma$. Let C_n ($\neq \gamma+\widetilde{\gamma}$) denote a relative boundary of F_n and let $\gamma_n = C_n\cap\Omega$, $\widetilde{\gamma}_n = C_n\cap(F-\Omega)$ and $\Gamma_n=\Gamma\cap F_n$, $\Omega_n=\Omega\cap F_n$.

2. S and T operations. Methods and results in this section are due to Kuramochi who has solved affirmatively our unsolved problem II in our previous paper [1] and related problems. For completeness we shall explain his procedure with a slight modification.

Let $W(z)$ be any member of Q_Ω. Let $W^n(z)$ be a function bounded and harmonic on F_n-F_0 satisfying the following conditions: $W^n(z)=0$ for $\gamma+\widetilde{\gamma}+\widetilde{\gamma}_n$ and $=W(z)$ for γ_n. Then evidently $W^n(z)\geqq W(z)$ holds on Ω_n, and therefore this leads to a fact that

$$\frac{\partial}{\partial\nu}\left(W^n(z)-W(z)\right) \geqq 0 \quad \text{on } \gamma_n$$

and

$$\frac{\partial}{\partial\nu} W^n(z) \geqq 0 \quad \text{on } \widetilde{\gamma}_n+\gamma+\widetilde{\gamma}.$$

Hence we see that

$$\begin{aligned}\infty > M &= \int_\Gamma \frac{\partial}{\partial\nu} W(z)\,ds \\ &\geqq \int_{\Gamma_n} \frac{\partial}{\partial\nu} W(z)\,ds = -\int_{\gamma_n}\frac{\partial}{\partial\nu} W(z)\,ds \\ &\geqq -\int_{\gamma_n}\frac{\partial}{\partial\nu} W^n(z)\,ds = \int_{\gamma+\widetilde{\gamma}+\widetilde{\gamma}_n}\frac{\partial}{\partial\nu} W^n(z)\,ds \\ &> \int_{\gamma+\widetilde{\gamma}}\frac{\partial}{\partial\nu} W^n(z)\,ds .\end{aligned}$$

Moreover we see easily that $W^n(z)\geqq W^m(z)$, for $n>m$ on Ω_m. There-