ON EXCEPTIONAL VALUES OF A SOLUTION OF A DIFFERENTIAL EQUATION

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Consider a differential equation

$$(1)\quad P_0(x)y'' + P_1(x)y' + P_2(x)y = P_3(x)$$

where $P_0(x)$, $P_1(x)$, $P_2(x)$ and $P_3(x)$ are polynomials of x.

W.Gross[1)] proved that the exceptional values of any non-constant analytic solution of the above equation cannot cover any continuum of values.

In the present paper, we shall show that the term "any continuum of values" in Gross' theorem can be replaced by "any set of positive capacity".

1. Let $a_1, a_2, \ldots, a_k$ be the zeros of $P_0(x)$. It is well known that any solution of (1) cannot have a singularity besides the above points $a_1, a_2, \ldots, a_k$ and $a_0 = \infty$.

We first uniformize a solution $y(x)$. Consider namely a Fuchsian function $\varphi(t)$ with a fundamental domain B as follows: all the vertices of B are parabolic double points and are situated on the unit circle, and $\varphi(t)$ is constructed such that it tends to one of the points a_i when t approaches through inside of B to any one of the above vertices. Moreover no solution of the equation $\varphi(t) = a_i$ lies in the interior of the unit circle $|t| < 1$.

Now, put $x = \varphi(t)$. If x turns around one of the points a_i, the corresponding t cannot return to its initial value, as the corresponding vertex is a parabolic point. Hence if we consider the solution $y(x)$ as a function of t, it becomes one-valued in $|t| < 1$, a fact which is also evident from the monodromy theorem. Let this function be denoted by $y = y(\varphi(t)) \equiv \psi(t)$.

2. Suppose that the set of exceptional values of $y(x)$ were of capacity positive, and denote it by E. Choose any three points of E, α, β and γ, say. Now consider a triangle function $\mu = \lambda(y)$ which maps the interior of the circle passing through α, β, and γ onto the interior of a curvilinear triangle whose sides consist of three circles, each of which intersects the unit circumference perpendicularly, and whose vertices are the images of α, β and γ. Obviously the inverse function $y = \lambda^{-1}(\mu)$ is one-valued and has the unit circumference $|\mu| = 1$ as its natural boundary. Let E_μ be the image of E on the μ-plane. Then the capacity of E_μ is positive.

Consider $\mu = \lambda(y) = \lambda(\psi(t)) \equiv \mu(t)$ as a function of t. As Gross has proved, if y tends to any one of accessible boundary points of E, the corresponding x must converge to one of the points a_i. Hence no image of E on the t-plane lies in the interior of the unit circle, $|t| < 1$.

3. Let $t(\mu)$ be the inverse function of $\mu(t)$. The projection on the μ-plane of the Riemann surface of $t(\mu)$ belongs to the unit circle $|\mu| \leq 1$ and does not cover the set E_μ. Finally, let us consider the function $x = \varphi(t(\mu)) \equiv \xi(\mu)$ which is one-valued and holomorphic on the above Riemann surface. If μ tends to any accessible boundary point of E_μ which lies in $|\mu| < 1$, the corresponding y tends to one of the accessible boundary point of E, and therefore the corresponding $x = \xi(\mu)$ tends to one of the points a_i according to Gross' reasoning. As the set of the points a_i is finite, a theorem of Tsuji[2)] implies