ON THE ABSOLUTE SUMMABILITY FACTORS

By Gen-ichiro SUNOUCHI

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1. In the present note the author proves the following theorem which is an answer of the problem raised by M.T.Cheng [2].

Let $\varphi(t)$ be even, periodic, integral and

(1) $$\varphi(t) \sim \sum_{n=1}^{\infty} a_n \cos nt.$$

If we denote by $\varphi_\alpha(t)$ the α-th mean of $\varphi(t)$, then we have the following theorem.

Theorem 1. If $\varphi_\alpha(t)$ is bounded variation in $(0, \pi)$, then $\{\log(n+1)\}^{-(1+\varepsilon)}$ $(\varepsilon>0)$ are the $|C.\alpha|$-summability factors of the Fourier series of $\varphi(t)$ at $t=0$.

Cheng proved this theorem for $0 \leqq \alpha \leqq 1$, and said that the case $\alpha > 1$ remains open. But this theorem is a easy consequence of

Theorem 2. Denote by σ_n^α the $(C.\alpha)$-mean of the series $\sum a_n$. If

$$\sum_{n=1}^{N} |\sigma_n^\alpha - \sigma_{n-1}^\alpha| = O(\log N),$$

then $\{\log(n+1)\}^{-(1+\varepsilon)}$ are the $|C.\alpha|$-summability factors of the series $\sum a_n$.

Denote by $\sigma_n^\alpha(0)$ the $(C.\alpha)$-mean of the Fourier series of (1) at $t=0$. Then from Bosanque's theorem [1], if $\varphi_\alpha(t)$ is bounded variation in $(0,\pi)$,

$$\sum_{n=1}^{N} |\sigma_n^\alpha(0) - \sigma_{n-1}^\alpha(0)| = O(\log N).$$

From this fact if Theorem 2 is proved, Theorem 1 is evident.

2. Concerning Theorem 2, we shall raise the problem :

if $\sum_{n=1}^{N} |\sigma_n^\alpha - \sigma_{n-1}^\alpha| = O(\log N)$ and

$$\sum_{n=1}^{\infty} a_n/\log(n+1) \quad \text{is}$$

$|C|$-summable for some order, then whether $\{\log(n+1)\}^{-1}$ are the $|C.\alpha|$-summability factors or not. In the ordinary Cesaro summability case, this problem has been answered affirmatively by A.Zygmund [5], (cf. G.Sunouchi [4] and L.Jesmanowicz [3]). But in the $|C.\alpha|$ case, we cannot drop $\varepsilon\ (>0)$. For $\alpha=0$, there is a function of bounded variation where

$$\sum |a_n|/\log(n+1) = \infty.$$

3. We proceed the proof of Theorem 2. Put $\mu_n = (\log n)^{1+\varepsilon}$, then

(1) $$\Delta^j \frac{1}{\mu_n} = O\left\{\frac{1}{n^j(\log n)^{2+\varepsilon}}\right\},$$

for $j = 1,2,\ldots$

From Kogbetliantz's formula, we have

(2) $$\sigma_n^\alpha - \sigma_{n-1}^\alpha$$

$$= \frac{1}{nA_n^\alpha}\sum_{\nu=1}^{n} \nu A_{n-\nu}^{\alpha-1} a_\nu = \frac{1}{nA_n^\alpha} t_n^{\alpha-1},$$

say. Further put

(3) $$\tau_n^\alpha = \frac{1}{A_n^\alpha}\sum_{\nu=0}^{n} A_{n-\nu}^\alpha \frac{a_\nu}{\mu_\nu}$$

$$= \frac{1}{A_n^\alpha \mu_n}\sum_{\nu=0}^{n} A_{n-\nu}^\alpha a_\nu + \frac{1}{A_n^\alpha}\sum_{\nu=0}^{n} A_{n-\nu}^\alpha a_\nu \left(\frac{1}{\mu_\nu} - \frac{1}{\mu_n}\right)$$

$$= u_n^\alpha + v_n^\alpha,$$

then

$$\tau_n^\alpha - \tau_{n-1}^\alpha = u_n^\alpha - u_{n-1}^\alpha + v_n^\alpha - v_{n-1}^\alpha.$$

The last term