ON HARMONIC DIMENSION, II

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One of the most elegant systematic investigations for structure of a family of positive harmonic functions was made by R.S.Martin. He introduced the so-called Martin topology to the ideal boundary in the spatial case. M.Heins established many deep results concerning the ideal boundary of Riemann surfaces, though his subjects are very special.

In the present paper we shall explain a notion of harmonic dimension introduced in our previous paper in a somewhat clearer form.

1. First exposition.

Basic domain in the sequel is supposed to be a C-end or an extended C-end defined in our previous paper.

Let $F \in O_G$ and Ω be a subsurface with analytic curves Γ as its relative boundary which are not compact. Let $\hat{\Omega}$ denote the doubled surface of Ω, symmetric with regard to Γ, then $\hat{\Omega}$ belongs to O_G. (See Z. Kuramochi [1])

Moreover we see easily the following fact: Let Ω be the same as above. Let γ be a compact part of Γ. Let $\hat{\Omega}$ denote the doubled surface of Ω, symmetric with regard to $\Gamma - \gamma$, then $\hat{\Omega}$ can be imbedded in a Riemann surface belonging to O_G.

In our case Ω is a C-end or an extended C-end, therefore $\hat{\Omega}$ is an end in the sense of Heins whose boundary consists of $\gamma + \tilde{\gamma}$, $\tilde{\gamma}$ being the symmetric image of γ with respect to $\Gamma - \gamma$.

Now we assume that $\hat{\Omega}$ has finite harmonic dimension in the sense of Heins. Let $P_{\hat{\Omega}}$ be a family of positive harmonic functions vanishing on $\gamma + \tilde{\gamma}$. Then all the minimal positive harmonic functions introduced by R.S.Martin can be obtained from the Green function of $\hat{\Omega}$ by a limiting process along a suitably selected non-compact point-sequence consisting of the logarithmic poles $p_n^{(i)}$ of the Green function of $\hat{\Omega}$, that is,

$$\lim_{n\to\infty} G_{\hat{\Omega}}(z, p_n^{(i)}) = v_i(z).$$

Among these limit functions there are all the generators $(v_1, \cdots, v_m)$ of $P_{\hat{\Omega}}$.

Let $\tilde{z}$ be a symmetric point of z with respect to $\Gamma - \gamma$. Then we can select a set of generators $(v_1, \ldots, v_m)$ of $P_{\hat{\Omega}}$ such that, for any i, either

$$v_i(z) = v_i(\tilde{z})$$

or

$$v_i(z) = v_j(z), \quad i \neq j$$

hold for a suitable j and this correspondence $\{i\} \to \{j\}$ is one-to-one and onto manner as a whole. Here we define $\tilde{v}_j(z) = \lim_{n\to\infty} G(z, \tilde{p}_n^{(j)})$.

To see this, we proceed as follows. If $(v_1(z), \ldots, v_m(z))$ is a set of generators of $P_{\hat{\Omega}}$, then $(\tilde{v}_1(z), \ldots, \tilde{v}_m(z))$ is also so. In fact, from the symmetry character $v_i(\tilde{z}) = \tilde{v}_i(z)$, we have that

$$\sum_{i=1}^{m} c_i \tilde{v}_i(z) = 0$$

implies

$$\sum_{i=1}^{m} c_i v_i(z) = 0.$$

Hence all the c_i vanish, and

$$\tilde{v}(z) \equiv v(\tilde{z}) = \sum_{i=1}^{m} a_i v_i(z)$$

holds by the assumption whence we see that

$$v(z) = \sum_{i=1}^{m} a_i v_i(\tilde{z}) = \sum_{i=1}^{m} a_i \tilde{v}_i(z)$$

remains valid also. This shows that $(\tilde{v}_1(z), \ldots, \tilde{v}_m(z))$ is also a set of generators of $P_{\hat{\Omega}}$.