ON HARMONIC DIMENSION

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An important notion of harmonic dimension of an end has recently been introduced by M. Heins. He has exhaustively investigated the relation between harmonic dimension and bounded harmonic or analytic functions on an end.

For completeness we shall explain the concept of harmonic dimension of an end in the sense of Heins.

An admitted Riemann surface of finite or infinite genus subjects to the following two conditions:

(i) the surface has precisely one ideal boundary element,
(ii) the ideal boundary is null in the sense of R. Nevanlinna.

An end is a subregion of an admitted surface whose complement is compact. Without loss of generality we may assume that the relative boundary of an end consists of a finite number of compact smooth Jordan curves. We shall abbreviate such an end by H-end. Let Ω be an H-end and P_Ω denote the family of non-trivial functions non-negative, one-valued and harmonic on Ω which vanish continuously on the relative boundary.

Harmonic dimension of an H-end is then the minimum number of elements of P_Ω which generate P_Ω with non-negative coefficients provided that such a finite set exists, otherwise it is ∞. We shall abbreviate this integer or ∞ by $H(\Omega)$. Heins has also given an example of an end of an arbitrarily assigned finite $H(\Omega)$. He has stated further that the existence of an end of infinite $H(\Omega)$ seems plausible.

In the present paper we shall introduce another sort of harmonic dimension of a domain. For this purpose, we first state the known results concerning the behavior of Green function at a neighborhood of the ideal boundary. Because of various difficulties we cannot yet succeed to construct the theory in its fully general form. We shall finally offer various unsolved problems which would seem very important for the investigation of the structure of an ideal boundary as well as the classification theory.

1. C-end and Green function.

Admitted Riemann surfaces are the same as in Heins' case. Let Ω be a subsurface of an admitted surface satisfying the following conditions:

(i) Ω has a finite number of non-compact piecewise smooth Jordan curves as its boundary Γ,

(ii) $\Omega \cap (F-F_n)$ has only one component for any n, where F and F_n denote the original admitted surface and its n th exhausting domain, respectively.

This Ω is abbreviated as a C-end. Let $\Omega_n = \Omega \cap F_n$ being assumed to be connected, and $\Gamma_n = \Gamma \cap F_n$ and let γ_n be the remaining boundary of Ω_n.

Let $g(z,t)$ be the Green function of Ω with the pole at $t \in \Omega_1$. Then $g(z,t)$ is a harmonic function bounded on $\Omega - \Omega_n$, $n \geqq 3$. By maximum principle $\underset{\Omega-\Omega_n}{\mathrm{Max}}\, g(z,t) = \underset{\gamma_n}{\mathrm{Max}}\, g(z,t) \equiv M(t,n)$. Moreover $M(t,n) \geqq M(t,m)$ holds for $n < m$. Thus there happen two possibilities, that is, either $\lim_{n\to\infty} M(t,n) = 0$ or > 0. In the former case the ideal boundary element is called regular and in the latter case irregular with respect to Ω.

In the regular case $\lim_{m\to\infty} g(z_m,t)$ exists and vanishes for any non-