ON A LEMMA OF SUNOUCHI AND YANO

By Masahiro NAKAMURA

Consider a jumper on the circle with length one in which an interval J forms a snare. The jumper jumps width c at one jump where c is a certain fixed irrational number. It is clear, he will fall into the snare after some jumps wherever he starts. The number m of jumps by which he fades away will depend on his starting point a.

Recently, G. Sunouchi and S. Yano [1] proved in their note on the absolute convergence of a certain function series, to generalizing a theorem of Szász, a lemma which can be formulated in our terms as follows:

PROPOSITION. There exists a positive integer k such that the jumper fades away into the snare after at most k jumps wherever he starts.

In this note, it will be shown that the proposition remains true even if the circle is replaced by certain abstract spaces.

A group G is called monothetic if G is compact abelian and if G contains an element c whose iterations nc are dense in G. Clearly, the torus is a monothetic group with an irrational number as generator. In this case, an open set N plays a role of the snare and the transformation f which maps x to $s+c$ can be considered as the jump in G. If the proposition is proved in this case, it is a slight generalization of Sunouchi-Yano's lemma.

Since G is monothetic and f is topological, the set A of all elements with the form $a+nc$ is also dense in G, for an arbitrary starting point a. Using this fact, we shall generalize the situation.

Let S be a compact Hausdorff space, and let T be a homeomorphism on S. T will be called a jump if the set of all $T^n(a)$'s (with $n = 1, 2, 3, \cdots$) is dense in S whenever a is a point of S which will be called starting point. An open set N of S will be called a snare. A jumper starting from a is called that he fades away after n jumps provided n is the least positive integer of the solution of the following implication:

$$T^n(a) \in N.$$

We shall prove the proposition under these circumstances. Clearly, our former monothetic case are contained in it.

PROOF: A point x will be called a point of the (first) scoring position N_1 if $T(x)$ is contained in N. Since T is topological, the scoring position N_1 is an open set. The second scoring position N_2 is defined as the set of all points which can reach the first scoring position after a jump. The n-th scoring position N_n will be defined by the induction. It is clear that N_n is open too. (For the sake of the simplicity, we may assume that a point can belong to two or more scoring positions, i. e., the jumper continues imaginary jumps after he fell into the snare.)

It is obvious that the jumper starting from a fades away after at most n jumps if and only if a belongs to the set union $N_1 \cup N_2 \cup \dots \cup N_n$. Hence, to prove the proposition, it suffices to show that the finite number of scoring positions covers the space:

$$S = \bigcup_{i=1}^{k} N_i .$$

The maximum of the indices of such scoring positions becomes our k in the proposition.

By the compactness of the space S, it suffices to insure the above statement to prove that the space S is covered by all scoring positions, i. e.,