ON THE CONVERGENCE OF A MULTIPLE POWER SERIES

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§1. The notion of convergence of a multiple series is somewhat complicated; we can consider several kinds of convergences. Here we treat mainly double sequences or double series for simplicity's sake, but the same holds for general multiple ones.

Usually the convergence of a double sequence s_{mn} is defined as follows: a double sequence

$\{s_{mn}\}_{m,n=0}^{\infty}$ converges to s, if for any given $\varepsilon > 0$, we can find a number $\ell_0 = \ell_0(\varepsilon)$ such that for every $m, n \geqq \ell_0$, we have $|s_{mn} - s| < \varepsilon$. The convergence of a double series

$$(1) \qquad \sum_{m,n=0}^{\infty} a_{mn}$$

whose sum is s, is defined by the convergence of its partial sums

$$(2) \qquad s_{mn} \equiv \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} a_{\mu\nu}$$

to s in the above sense. We call this the <u>P-convergence</u> (P means the "partial sum") in this paper.

On the other hand, we say a double series (1) is <u>A-convergent</u> (A means the "arrangement"), if at least one of the simple series in which the original series has been arranged is convergent.[1] In this case, the sum has no mean generally, because it depends on the arrangement, unless (1) is absolutely convergent.

It is evident that these two notions coincide with each other for the series with positive terms, and that the absolutely convergent series is A- and P- convergent. But a series which is A- and P-convergent is not always absolutely convergent as easily shown by:

<u>Example 1.</u>

$$\begin{array}{l} 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \cdots \\ +(-1)+(-\tfrac{1}{2})+(-\tfrac{1}{3})+(-\tfrac{1}{4}) + \cdots \\ +\ 0 + 0 + 0 + 0 + \cdots \\ +\ 0 + 0 + 0 + 0 + \cdots \\ + \cdots\cdots\cdots \end{array}$$

Also these two convergences are not the same in general cases. In fact:

<u>Example 2.</u>

$$\begin{array}{l} 0 + 1 + \tfrac{1}{3} + \tfrac{1}{5} + \cdots \\ +(-\tfrac{1}{2}) + 0 + 0 + 0 + \cdots \\ +(-\tfrac{1}{4}) + 0 + 0 + 0 + \cdots \\ +(-\tfrac{1}{6}) + 0 + 0 + 0 + \cdots \\ + \cdots\cdots\cdots \end{array}$$

is A-convergent but not P-convergent. Conversely,

<u>Example 3.</u>

$$\begin{array}{l} -1 + 1 + 2 + 3 + 4 + \cdots \\ +1 +(-1)+(-2)+(-3)+(-4)+ \cdots \\ +2 +(-2) + 0 + 0 + 0 + \cdots \\ +3 +(-3) + 0 + 0 + 0 + \cdots \\ + \cdots\cdots\cdots \end{array}$$

is P-convergent but not A-convergent.

We remark that the terms of an A-convergent series are bounded, but this is not true for P-convergent series as has already been shown in Example 3. However, we have from the definition,

<u>Lemma 1.</u> If the double series

$$(1) \qquad \sum_{m,n=0}^{\infty} a_{mn}$$

is P-convergent, there exists a number ℓ such that its partial sums

$$(2) \qquad s_{mn} = \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} a_{\mu\nu}$$

are uniformly bounded for m, n, provided that both suffixes m and n are $\geqq \ell$.

<u>Corollary.</u> Since we have

$$(3) \qquad a_{mn} = s_{mn} - s_{m\,n-1} - s_{m-1\,n} + s_{m-1\,n-1} \quad (m, n \geqq 1),$$