ON THE STRONG LAW OF LARGE NUMBERS

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Let $X_1, X_2, \dots, X_n, \dots$ be a sequence of independent random variables and let the mean of X_n, $E(X_n)=0$, $n=1,2,\dots$. If

$$(1)\quad \frac{S_n}{n} = \frac{X_1+X_2+\dots+X_n}{n}$$

converges to zero with probability 1, we say that the sequence $\{X_n\}$ obeys the strong law of large numbers.

Sufficient conditions for the vality of the strong law of large numbers were given by various authors. Recently H.D.Brunk[1] has given the extension of the Kolmogoroff's sufficient condition [2] when each random variable X_n have higher moments than the second order and has proved that:

If $E(X_n)=0$, $(n=1,2,\dots)$

$$(2)\quad \sum_n b_n^{(2q)}/n^{q+1}$$

converges for some positive integer q, then the sequence $\{X_n\}$ obeys the strong law, where

$$b_n^{(2q)} = E(X_n^{2q}),\ n=1,2,\dots.$$

More generally he has shown the following theorem.

Let $\{p_n\}$ be a sequence of positive constants, increasing to infinity such that

$$(3)\quad \liminf_{n\to\infty}(p_{n+1}-p_n)=h>0,$$

and (4) $\quad p_{n+1}/p_n < R, \quad (n=1,2,\dots)$

for some positive constant R, then if

$$E(X_n)=0 \quad (n=1,2,\dots)$$

and (5) $\quad \sum b_n^{(2q)}/p_n^{q+1}$

converges for some positive integer q, then

$$(6)\quad \frac{S_n}{p_n} = \frac{X_1+X_2+\dots+X_n}{p_n}$$

converges to zero with probability 1.

We shall give simple proofs and slight generalizations of these theorems appealing to an inequality theorem of Marcinkiewicz and Zygmund[3][4] and to a theorem due to one of the authors[5] which is quoted as:

Lemma 1. For any positive ε, let

$$(7)\quad P_r\left\{\varepsilon > \frac{S_n}{n} > -\varepsilon\right\} \geq 1-\delta_n(\varepsilon),$$

$$\delta_n(\varepsilon)\to 0,\quad (n\to\infty)$$

and suppose that for any $\varepsilon>0$

$$(8)\quad \sum_{k=1}^{\infty}\delta_{2^k}(\varepsilon) < \infty.$$

Then the sequence $\{X_n\}$ obeys the strong law of large numbers.

We restate the theorem, in which q does not need to be an integer.

Theorem 1. If $E(X_n)=0$ $(n=1,2,\dots)$ and

$$(9)\quad \sum_{n=1}^{\infty}\frac{b_n^{(q)}}{n^{\frac{q}{2}+1}}$$

converges for some real q, $q \geq 2$, then the sequence $\{X_n\}$ obeys the strong law of large numbers, where $b_n^{(q)} = E(|X_n|^q)$, $n=1,2,\dots$.

Proof of Theorem 1. Let

$$P_r\{|S_n| > n\varepsilon\} = \delta_n(\varepsilon).$$

Then by Lemma 1, it is sufficient to prove

$$\sum_{k=1}^{\infty}\delta_{2^k}(\varepsilon) < \infty, \quad \text{for any } \varepsilon>0.$$

If we put $q = 2r$, then $r \geq 1$. By a theorem of Marcinkiewicz and Zygmund [3],

$$E(|S|^{2r}) \leq A_q E((X_1^2+X_2^2+\dots+X_n^2)^r),$$

where A_q is an absolute constant which depends only on q.

By Holder's inequality

$$E((X_1^2+X_2^2+\dots+X_n^2)^r) \leq n^{r/r'}\sum_{k=1}^{n} b_k^{(2r)},$$

$$\frac{1}{r}+\frac{1}{r'}=1.$$

Thus Tcheby cheff inequality shows

$$P_r\{|S_n| \geq n\varepsilon\} \leq (n\varepsilon)^{-2r}E(|S_n|^{2r}) \leq A_q(n\varepsilon)^{-2r}n^{\frac{r}{r'}}\sum_{k=1}^{n} b_k^{(2r)}.$$

Hence

$$\delta_{2^k}(\varepsilon) \leq A_q\,\varepsilon^{-2r}2^{-k(r+1)}\sum_{i=1}^{2^k} b_i^{(2r)}.$$

Thus

$$\sum_{1}^{\infty}\delta_{2^k}(\varepsilon) \leq A_q\varepsilon^{-2r}\sum_{k=1}^{\infty}\frac{1}{2^{k(r+1)}}\sum_{i=1}^{2^k} b_i^{(2r)}$$