NOTE ON LAPLACE-TRANSFORMS, (II)

ON SOME CLASS OF LAPLACE-TRANSFORMS, (I)

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(1) THEOREM I. We consider the Laplace-transform

$$(1.1) \qquad F(s) = \int_0^\infty \exp(-sx) f(x)\,dx \quad (s=\sigma+it)$$

where $f(x)$ is R-integrable in any finite interval $0 \leqq x \leqq X$, X being an arbitrary positive constant. In general, $F(s)$ has three convergence-abscisses, i.e. simple convergence-abscissa σ_s, uniform convergence-abscissa σ_u, and absolute convergence-abscissa σ_a ($\sigma_s \leqq \sigma_u \leqq \sigma_a$), whose existence is well-known ([1] p.16, p.42 - See references placed at the end -). In the present Note, we discuss the sufficient conditions for $\sigma_s = \sigma_u = \sigma_a$. We begin with some definitions:

DEFINITION I. The sequence of intervals $\{I_\nu\}$ $(\nu=1,2,\cdots\cdots)$ is defined as follows:

$$\begin{cases} (i) & I_\nu : \quad t_\nu - \varepsilon(t_\nu) \leqq t \leqq t_\nu + \varepsilon(t_\nu), \\ & \qquad 0 < t_\nu \uparrow \infty, \\ (ii) & \lim_{\nu\to\infty} \varepsilon(t_\nu) = 0, \quad \lim_{\nu\to\infty} \frac{1}{t_\nu} \log\left(\frac{1}{\varepsilon(t_\nu)}\right) = 0. \end{cases}$$

DEFINITION II. We say that $f(t)$ belongs to the class $C\{I_\nu\}$, provided that

$$\begin{cases} (i) & \overline{\lim_{t\to\infty}} \frac{1}{t} \log|f(t)| = \lim_{\substack{t\to\infty \\ t\in\{I_\nu\}}} \frac{1}{t}\log|f(t)| \\ & = \alpha < +\infty, \\ (ii) & f(t) \text{ is continuous in } \{I_\nu\}, \\ (iii) & f_1(t) = \Im[f(t)] \text{ is differentiable in } \{I_\nu\} \text{ and} \\ & \overline{\lim_{\substack{t\to\infty \\ t\in\{I_\nu\}}}} \frac{1}{t}\cdot \log|f_1'(t)| \leqq \alpha. \end{cases}$$

Now we can establish

THEOREM I. If $f(t)$ is R-integrable in any finite interval and belongs to $C\{I_\nu\}$, then

$$\sigma_s = \sigma_u = \sigma_a = \alpha.$$

As an immediate consequence of Theorem 1, follows

COROLLARY. If $f(t)$ is real and continuous in $0 \leqq t < +\infty$, and $\lim_{t\to\infty} \frac{1}{t}\log|f(t)| = \alpha$, then $\sigma_s = \sigma_u = \sigma_a = \alpha$.

In order to prove Theorem 1, we need next Lemma:

LEMMA.

$$\beta = \overline{\lim_{t\to\infty}} \frac{1}{t}\log|F(t;\varepsilon(t))|$$
$$\leqq \sigma_s \leqq \sigma_u \leqq \sigma_a \leqq$$
$$\overline{\lim_{t\to\infty}} \frac{1}{t}\log|f(t)| = \alpha,$$

where

$$\begin{cases} (i) & F(t;\varepsilon(t)) = \frac{1}{2\varepsilon(t)} \int_{t-\varepsilon(t)}^{t+\varepsilon(t)} f(x)\,dx, \\ (ii) & \lim_{t\to\infty} \varepsilon(t) = 0, \\ & \lim_{t\to\infty} \frac{1}{t}\log\left[\frac{1}{\varepsilon(t)}\right] = 0 \end{cases}$$

Proof. By the definition of α, for any given ε (>0), there exists $T(\varepsilon)$ such that

$$|f(t)| < \exp((\alpha+\varepsilon)t) \quad \text{for} \quad t > T(\varepsilon).$$

Hence, denoting by $[t]$, as usual, the greatest integer contained in t, we have, for $[t] > T(\varepsilon)$,

$$\int_{[t]}^{t} |f(x)|\,dx < (t-[t]) \exp((\alpha+\varepsilon)t)$$
$$< \exp((\alpha+\varepsilon)t).$$

Accordingly, by K.Kurosu's formula ([2], [3]),

$$\sigma_a = \overline{\lim_{t\to\infty}} \frac{1}{t}\log\left(\int_{[t]}^{t} |f(x)|\,dx\right) < (\alpha+\varepsilon).$$

Letting $\varepsilon \to 0$,

$$(1.2) \qquad \sigma_a \leqq \alpha$$

By K.Kurosu's formula, we have

$$\sigma_s = \overline{\lim_{t\to\infty}} \frac{1}{t}\log\left|\int_{[t]}^{t} f(x)\,dx\right|$$
$$= \overline{\lim_{t\to\infty}} \frac{1}{[t]}\log\left|\int_{[t]}^{t} f(x)\,dx\right|.$$