ON THE COMMUTATIVITY OF THE C^*-ALGEBRA

By Takasi TURUMARU

(Communicated by T. Kawata)

Let R be a concrete C^*-algebra in the sense of I.E.Segal, and A be the totality of all self-adjoint elements of R. For x, y of A, define the formal product

$$x \circ y = (xy + yx)/2$$

then for every x, y, z of A, and for every real scalar α, we have

$$(\alpha x) \circ y = \alpha(x \circ y), \quad x \circ y = y \circ x,$$

and

$$(x + y) \circ z = (x \circ z) + (y \circ z).$$

Moreover, if R is commutative, then the associative law (*)

$$(x \circ y) \circ z = x \circ (y \circ z)$$

holds in A. In this note, we shall prove the converse

Theorem. If the associative law (*) is satisfied in A, then R is commutative.

Proof. Substituting $y = xz + zx$ in (*), we have $xz^2x = zx^2z$ for every x, z of A.

Let

$$x = \int \lambda de_\lambda, \quad y = \int \mu de'_\mu$$

be the spectral representations of x and y respectively. Then from the well known fact, the commutativity of the product xy is equivalent to that of $e_\lambda e'_\mu$ for all λ, μ. Moreover by a theorem due to J. von Neumann,(1) e_λ is the strong limit of a sequence from A, for every λ, so that, for every fixed λ, μ, we get two sequences $\{x_n\}$ and $\{y_n\}$ such that

$$\operatorname{strong\,lim}_{n\to\infty} x_n = e_\lambda,$$
$$\operatorname{strong\,lim}_{m\to\infty} y_m = e'_\mu;$$
$$x_n \in A, \; y_m \in A.$$

Hence

$$\operatorname{strong\,lim}_{m\to\infty} \operatorname{strong\,lim}_{n\to\infty} x_n y_m x_n = e_\lambda e'^2_\mu e_\lambda = e_\lambda e'_\mu e_\lambda$$

and

$$\operatorname{strong\,lim}_{m\to\infty} \operatorname{strong\,lim}_{n\to\infty} y_m x_n y_m = e'_\mu e^2_\lambda e'_\mu = e'_\mu e_\lambda e'_\mu.$$

On the other hand we have

$$x_n y_m x_n = y_m x_n y_m$$

for every m, n; therefore

$$e_\lambda e'_\mu e_\lambda = e'_\mu e_\lambda e'_\mu.$$

Set now $u = e_\lambda e'_\mu - e'_\mu e_\lambda$, then

$$\begin{aligned} uu &= (e_\lambda e'_\mu - e'_\mu e_\lambda)(e'_\mu e_\lambda - e_\lambda e'_\mu) \\ &= e_\lambda e'_\mu e_\lambda - e_\lambda e'_\mu e_\lambda e'_\mu - \\ &\quad - e'_\mu e_\lambda e'_\mu e_\lambda + e'_\mu e_\lambda e'_\mu. \\ &= 0, \end{aligned}$$

so $u = 0$; that is, $e_\lambda e'_\mu = e'_\mu e_\lambda$ for every λ, μ. Thus we get

$$xy = yx.$$

(*) Received May 24, 1951.

(1) J.von Neumann: Zur Algebra der Funktionaloperationen und Theorie der normalen Operatoren, Math. Ann., 102(1927) pp.370-427, especially 391-2.

Tôhoku University, Sendai.