A REMARK ON REGULARLY CONVEX SETS

By Jyôji HOTTA

(Communicated by Y. Komatu)

Let E be a normed (not necessarily complete) linear space and E^* its conjugate space. Then the regularly convex sets in E^* are defined by M. Krein and V. Šmulian as follows: a set $K^* \subset E^*$ will be called regularly convex if for every $f_0 \in E^*$ not belonging to K^* there exists an element $x_0 \in E$ such that

$$\sup_{f \in K^*} f(x_0) < f_0(x_0).^{1)}$$

This conception was studied in detail in their paper, but as they did not consider it on the standpoint of weak topology, their beautiful results were mainly restricted in separable case. Therefore we shall intend to simplify some of their results in general case. After that we shall give the proof of M. Krein and D. Milman's theorem on the existence of extreme points by means of Zorn's lemma.[2)]

Theorem 1. Every regularly convex set K^* in E^* is convex and weakly closed.

Let g, $h \in K^*$ and α, β satisfy the condition

$$\alpha > 0, \quad \beta > 0, \quad \alpha + \beta = 1.$$

If $\alpha g + \beta h$ were not contained in K^*, there should exist an element $x_0 \in E$ such that

$$\sup_{f \in K^*} f(x_0) < \alpha g(x_0) + \beta h(x_0).$$

On the other hand $g(x_0)$ and $h(x_0) \leq \sup_{f \in K^*} f(x_0)$, and hence

$$\alpha g(x_0) + \beta h(x_0) \leq (\alpha + \beta) \sup_{f \in K^*} f(x_0) = \sup_{f \in K^*} f(x_0).$$

This is a contradiction, and it means K^* is convex.

Next we shall show that K^* is weakly closed. Let $f_0 \in E^*$ be a limiting point of K^* in the weak topology and not contained in K^*. Then we can select an element $x_0 \in E$ such that

$$\sup_{f \in K^*} f(x_0) < f_0(x_0),$$

and hence if we choose a positive number $\varepsilon > 0$ satisfying the condition

$$f_0(x_0) - \sup_{f \in K^*} f(x_0) > \varepsilon,$$

Then the weak neighbourhood $U(f_0; x_0; \varepsilon)$ contains no elements of K^*. Therefore K^* must be weakly closed.

The next theorem which we shall prove in this paper is the inverse of the theorem 1, and as we mentioned above it has already been proved by M. Krein and V. Smulian when E is a separable space.[3)]

For this purpose we shall state the next two lemmas.

Lemma 1. Let $F_0(f)$ be a weakly continuous linear functional on E^*. Then there exist an element $x_0 \in E$ such that

$$F_0(f) = f(x_0)$$

for all $f \in E^*$.

This lemma has been proved by Banach[4)] making use of regularly closed set when E is separable, and that proof is also applied to general case if we notice to the fact that a linear subspace in E^* is regularly closed if and only if it is weakly closed. But we shall give here a direct proof without use of regularly closed set.

From the assumption $\{ f ;\ f \in E^*,\ |F_0(f)| < 1 \}$ is an open set in the weak topology containing the zero functional 0, hence there exist a finite number of elements x_1, x_2, ..., x_n of E such that

$$U(0; x_1, x_2, \dots, x_n; 1) \subset \{ f ;\ f \in E^*,\ |F_0(f)| < 1 \},$$